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AN EXAMINATION OF
CIRCULAR ERROR PROBABLE
APPROXIMATION TECHNIQUES

THESIS

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THESIS

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Preface

This report examines circular error probable (CEP) approximation methodologies. The approximations and methodologies explored in this report use statistical distributions and mathematical methods that may be unfamiliar to some people. It is assumed that the reader has a general understanding of statistical distributions, and their associated parameters as well as how to manipulate these distributions.

This thesis was suggested by persons from the 4220th Weapon System Evaluation Squadron, Strategic Air Command and should be useful to all offices interested in calculating CEPs for missiles, rockets, bombs, bullets, etc.

Before I begin this report, I would like to acknowledge some people without whom this thesis would never have been accomplished. Thanks go to Captains Paul Auclair and Dave Berg for sponsoring this effort and furnishing me with much of the information that was necessary to start and complete this study - I hope they find it useful. Also I would like to thank Major Bill Rowell, my faculty advisor, for believing in this topic and for his patience, criticism, help, and support. I thank my parents for always having believed in me and given me the foundation on which all else is built. Finally I would like to thank my fiancee, Gileen Gleason, for her unwavering support especially during the last month.

Richard L. Elder

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Abstract

Several approximation techniques are currently used to estimate CEP. These techniques are statistical in nature, based on the bivariate normal distribution of the crossrange and downrange miss distances of sample impacts of weapon systems. This thesis examines four of the most widely used approximation techniques (Grubbs-Patnaik/chi-square, Grubbs-Patnaik/Wilson-Hilferty, modified RAND-234, and correlated bivariate normal), compares their results with the results and computational effort required by established numerical integration techniques, determines the relative accuracy of each technique in various regimes of the bias/ellipticity parameter space. Included in this report is a tutorial on the subject of CEP meant to serve as a general introduction to how to calculate CEPs with some of the popular approximation techniques.

In general it was found that each of the approximation techniques is best in some regime of the parameter space with the Grubbs-Patnaik/chi-square technique being the most reliable estimator. For fast calculations of CEP, the correlated bivariate normal and the "exact" method may not be feasible because both are computationally rigorous and require from 2 minutes to several hours of computer time (on a personal computer) to give an estimate of CEP.

EXAMINATION OF CIRCULAR ERROR PROBABLE APPROXIMATION TECHNIQUES

I. Introduction

Background

Circular error probable (CEP) is defined as "... the radius of a circle centered at the target or mean point of impact within which the probability of impact is 0.5" (16:1). That is:

$$\iint_{C_{CEP}} F(x,y)\, dx\, dy = 0.5 \qquad (1.1)$$

where

C_{CEP} = a circle centered at the target with radius CEP

$F(x,y)$ is the bivariate normal density function

x is the crossrange miss distance

y is the downrange miss distance.

Given this definition, circular error probable (CEP) is used to measure a weapon system's impact accuracy. The CEP is determined by taking test impact data and approximating the actual CEP of the missile system. Of course, the more data points - or weapon system tests - collected, the better the approximation of CEP.

There are several ways to approximate CEP. One group of techniques is non-parametric approximation methods. These approximation techniques make no assumptions about the

underlying distributions of the impacts. One non-parametric method simply uses the sample median as an estimate of CEP. Non-parametric estimations are only useful in cases where one has available a large number (greater than 30) of sample impacts. When flight testing missiles it is not practical to expect to have a large number of data points (usually no more than 15 are available) (2), so non-parametric tests are of limited use and are not presented in this study.

The most common parametric methods for estimating CEP are based on the assumption that the impacts are normally distributed: closed form integration of the bivariate normal function, numerical integration of the bivariate normal, or algebraic approximations of the bivariate normal function (16:2-4). Closed form integration of the bivariate normal distribution is of limited use because it can only be accomplished for the case of non-correlated samples with means of zero and equal standard deviations (16:2-3).

On the other hand, numerical integration techniques are useful in estimating correlated samples with non-zero means and unequal variances. Numerical integration techniques yield a probability of impact given a circle of known radius. However in approximating CEP it is of interest to derive a radius given a known probability (i.e. 0.5). Therefore one must come up with an initial estimate of CEP and then iterate the numerical integration until a

probability of 0.5 is achieved (3).

It is important to note that numerical integration techniques for estimating CEP are in fact estimates of CEP. They are the most accurate estimates available under the assumption of normality. Because of the fact that they are most accurate they are often referred to as "exact" solutions. Although the use of an "exact" method provides the most accurate results, there are advantages and disadvantages to its use.

> While numerical integration methods provide good estimates of CEP, and can be used to evaluate the accuracy of the other CEP approximation methods, they require considerable computer time, and are usually impractical in flight test analysis (16:3).

The key point is that numerical integration can be used as a standard to evaluate the accuracy of other CEP approximation methods.

The preferred method of approximating CEP using numerical integration of the bivariate normal density function is infinite series expansion. When calculating CEP, one is solving for the radius within which the probability of hit is 0.50.

$$P(R) = \iint \frac{1}{2\pi S_x S_y} \left\{ \exp\left[-\frac{1}{2} \left(\frac{(x - \bar{x})^2}{S_x^{\,2}} + \frac{(y - \bar{y})^2}{S_y^{\,2}} \right) \right] \right\} dxdy \quad (1.2)$$

where,

(x, y) = downrange and crossrange miss distances;

$(\bar{x}, \bar{y})$ = downrange and crossrange sample means;

(S_x, S_y) = downrange and crossrange sample standard deviations.

When P(R) is equal to 0.50, then R is the estimate of the CEP. Smith suggests an iterative approach to solve for CEP (17:1-6). One approximates this integral (1.2) with an infinite series and expands it until the inner terms approach zero. This method of approximating CEP is referred to as the "exact" method. The calculations required by the "exact" method can be extremely rigorous as will be illustrated in Chapter III. The number of iterations for the series to converge can use much computer time.

Another numerical integration technique in current use is the correlated bivariate normal method (CBN). The CBN is sometimes a faster approximation than the "exact" method, however, it also requires considerable time to converge to the CEP.

Since numerical integration techniques may be impractical, it is essential to have fast approximation techniques that are fully tested and validated. The algebraic approximations give fast results and are accurate over certain regimes of the parameter space (Three parameters can be used to characterize the probability distributions of impacts: bias, ellipticity, and correlation. It will be shown that correlation can be removed from sample data so that one is left with two parameters). Many different algebraic approximations of CEP

exist. Three of these algebraic approximation methods are of particular interest to this study of CEP methodologies:

Grubbs-Patnaik/chi-squared
Modified RAND-234
Grubbs-Patnaik/Wilson-Hilferty

These three methods were suggested by the sponsors of this study (4220th Weapon System Evaluation Squadron, Strategic Air Command) as the most frequent methods used. No work has been accomplished to fully compare these approximations with each other and with the numerical integration methods.

Problem Statement

Given non-correlated sample impacts, how do common CEP approximation techniques (Grubbs-Patnaik/chi-squared, Modified RAND-234, Grubbs-Patnaik/Wilson-Hilferty, or CEN) compare in accuracy and computational effort (measured by computer time) to the "exact" method (numerical integration) over the possible range of the parameters bias and ellipticity.

Purpose of the Study

It has been suggested that the "exact" numerical integration method of approximating circular error probable (CEP) is not practical for flight test analysis because of the inordinate amount of computer time it takes for this method to produce an approximation of CEP (16:3). This study examines four methods of approximating CEP and compares them to the "exact" method.

While examining these approximation methods this study

determines which method gives the "best" estimate of CEP when compared to the "exact" methodin various regimes of the bias/ellipticity parameter space. Additionally, a tutorial on CEP and the various approximation techniques for calculating CEP is presented.

The general approach to this thesis is to:

-- Apply the three algebraic approximation techniques and the CBN over a wide range of the bias/ellipticity parameter space;

-- Compare accuracies and computational effort (computer time) of each of the four approximation techniques with the "exact" numerical integration method;

-- Analyze results to determine:

  - which technique is most accurate (compared to the "exact" method) in a given regime of the bias/ellipticity parameter space;

  - where any of the techniques may fail to give an accurate estimate of CEP.

-- Use regression analysis to estimate the error generated by each approximation technique over the bias/ellipticity parameter space and add a correction factor to the calculations.

This study does not use actual test data for validating the approximation techniques. Non-dimensional values of bias and ellipticity are used. Therefore, correlation is

assumed to be equal to zero. Given actual test data, the correlation can be removed from the data by translating the data to principal axes (See Appendix B for details on how to remove correlation from sample data) (10:20-21). The units (feet, meters, etc.) for CEP or any of the parameters do not affect the calculation of CEP, as long as one is consistent with the units used.

Sequence of Presentation

Chapter II presents a review of related literature in this field. Chapter III describes the three algebraic approximations, the CBN, and the "exact" method. Also Chapter III explains the data collection process, computer program and the regression analysis used to estimate the errors between the algebraic approximations and the "exact" solution.

Chapter IV discusses the results of this study in terms of which approximations of CEP are "best," and in what regimes of the bias/ellipticity parameter space each should be used. The results of the regression analysis are also discussed in Chapter IV. Chapter V summarizes the findings of this study and makes recommendations for the implementation of these results and for further research.

The appendices present the tutorial on circular error probable, the derivation of some of the mathematical formulae used in this study, computer program listings, and some sample output. Appendix A contains the tutorial on

CEP. This tutorial is written so that it can stand alone from the rest of this document. Appendix B is a discussion of the method of removing the correlation in the samples of impact data. Appendix C is a discussion of the secant method that is used in the numerical integration techniques. Appendix D contains sample results from the data collected for this report. Finally, appendix E is a listing of the computer program written in Pascal.

II. Literature Review

Past work in the area of approximating circular error probable can be divided into two groups: theoretical work and applied work.

Theoretical Work

The theoretical work in the area of estimating CEP is concerned primarily with deriving the various approximation techniques. Chapter III will give the details about the derivations of each approximation method.

The theoretical groundwork for two of the approximation techniques of interest to this study was set by Frank E. Grubbs. He combined the work of Patnaik, Wilson, and Hilferty to formulate two methods of approximating CEP. These are referred to as the Grubbs-Patnaik/chi-square, and the Grubbs-Patnaik/Wilson Hilferty (7). Grubbs defined the problem of interest as:

> that of finding the probability of hitting a circular target ... whether the delivery errors are equal or unequal and also for point of aim or center of impact of the rounds either coinciding with the target centroid or offset from it. ... It therefore appears desirable to record a straight-forward, unique, and rather simple technique for approximating probabilities of hitting for all of the various cases referred to above (7:51).

Grubbs' methods are being used by several organizations to calculate CEP: Strategic Air Command (missiles and bombers) (4), the Army Missile Command (18), and USAF Foreign Technology Division to name a few.

In addition to Grubbs' two approximation methods the

Rand-234 method is presented. This method is not based on Grubbs' work. The RAND-234 method of approximating CEP is simply a mathematical expression derived to approximate RAND-234 tables of probabilities (14). The RAND-234 tables were contained in RAND Report R-234.

> R-234 tables contain the probabilities of missing a circular target of a given radius with a weapon system of known accuracy aimed at some point offset from the center of the target. R-234 assumes that the weapon system accuracy can be described by a radially symmetric Gaussian distribution (14:2).

A least squares regression was used to derive this formula. The reason that this method was developed was that "... a formula facilitates computer calculation of CEP and obviates the need to look up CEPs in the R-234 tables" (14:2). This regression resulted in a third order polynomial for calculating CEP.

Two other CEP approximation techniques developed in the literature are numerical integration techniques. One is called the correlated bivariate normal (CBN), and the other is referred to as the "exact" method because it is considered the most accurate approximation (it is also the most computationally rigorous of the approximations).

The CBN was developed "... to calculate CEP about the target point given that the center of the distribution of impacts is located at some distance (range-track) defined by an impact bias vector" (15:1). The CBN solves for CEP by expressing the bivariate normal distribution in polar coordinates and then integrating over r and summing over θ.

The resulting expression must then be solved iteratively for the CEP value (r) that results in a probability of 0.5 (15:1-5). Typically 40 steps were used in this integration. Jones, however, notes the accuracy of the CBN can be improved by increasing the number of steps used in the integration (11:1).

Another numerical integration technique for approximating CEP is the "exact" method. The "exact" method is described in detail by Smith. This method involves Taylor series and infinite series expansions to estimate a bivariate normal distribution with zero correlation. After expansion, one must solve for the radius (r) within which the probability of hit is 0.5. This is done by iterating on r until the series expansion converges to 0.5. Whereas the CBN only involved summation over one variable (θ), the "exact" method sums over both r and θ (16).

Applied Work

Comparisons of some of these methods have been accomplished, but never has there been a study to compare all five of these approximation methods at the same time.

Smith compared the RAND-234, Grubbs-Patnaik/chi-square, and the Grubbs-Patnaik/Wilson-Hilferty methods with the infinite series, "exact" approximation (16). In this work Smith showed that one can produce a very significant error in calculating CEP by neglecting the rotation to principal axes in the case of correlated samples. This error was as

much as 23% in one case. Also Smith varied eccentricity while holding bias constant and vice versa. She did not, however show the effect of varying both bias and eccentricity at the same time (16:9).

Other works have also compared approximation methods. Jones compared the RAND-234, CBN (with 40 and 400 iterations), Grubbs-Patnaik/chi-square, and the "exact." Table 2-1 shows the values of ellipticity and bias used in the 16 cases examined by Jones.

Table 2-1

Cases (11:3)

Case	Ellipticity	Bias
1	0.25	0.5
2	0.30	0.6
3	0.35	0.7
4	0.40	0.8
5	0.45	0.9
6	0.50	1.0
7	0.55	1.1
8	0.60	1.2
9	0.65	1.3
10	0.70	1.4
11	0.75	1.5
12	0.80	1.6
13	0.85	1.7
14	0.90	1.8
15	0.95	1.9
16	1.00	2.0

(Note: the units in this analysis do not matter as long as one is consistent with the units -- once in meters, always in meters, etc.)

Table 2-2 shows the CEPs calculated for each of the techniques (RAND-234, CBN with 40 steps, CBN with 400 steps,

Grubbs-Patnaik/chi-square, and "exact") for each of the 16 cases in table 2.1. Table 2.2 illustrates that the "best" approximation varies depending on the values of ellipticity and bias. However, this paper did not explicitly delineate where one method becomes more inaccurate than another.

Table 2-2

Results (11:4)

Case	RAND	CBN(40)	CBN(400)	Grubbs	Exact
1	801.18	805.98	818.81	824.41	820.27
2	830.73	833.54	845.20	847.12	846.56
3	865.21	866.81	877.67	876.89	878.97
4	904.28	905.23	915.57	913.48	916.82
5	947.60	947.08	957.09	954.52	958.34
6	994.80	992.51	1002.37	999.71	1003.61
7	1045.55	1041.44	1051.28	1048.37	1052.55
8	1099.48	1093.69	1103.67	1100.02	1104.97
9	1156.25	1149.01	1159.28	1154.41	1160.62
10	1215.51	1207.01	1217.72	1211.11	1219.12
11	1276.90	1267.24	1278.54	1269.88	1280.01
12	1340.08	1329.22	1341.24	1330.47	1342.80
13	1404.69	1392.53	1405.40	1392.65	1407.04
14	1470.39	1458.07	1471.90	1473.64	1457.47
15	1536.82	1521.86	1536.68	1521.02	1538.52
16	1603.64	1587.43	1603.32	1586.85	1605.27

By calculating the error of each approximation relative to the "exact" method, one can determine which approximation is "best" for each case. Equation 2.1 is used to determine the relative error (RE) (16:9).

$$RE = \frac{CEP_{approx} - CEP_{exact}}{CEP_{exact}} \tag{2.1}$$

For example, table 2-3 contains relative errors for five of the example cases.

Table 2-3

Relative Errors

Case	RAND	CBN(40)	CBN(400)	Grubbs
1	-0.023	-0.017	-0.002	0.005
5	-0.011	-0.012	-0.001	-0.004
10	-0.003	-0.010	-0.001	-0.007
13	-0.002	-0.010	-0.001	-0.010
16	-0.001	-0.011	-0.001	-0.012

For all cases the CBN with 400 steps is the "best." However as stated before this method is computationally rigorous much like the "exact" method. So if one needs a "quick" estimate for case 1 the Grubbs-Patnaik/chi-square is the "best." However, by case 10 the RAND-234 gives the "best" estimate of the fast methods, and by case 16 the RAND-234 is as accurate as the CBN with 400 steps.

As mentioned before, no study has been accomplished to analyze the accuracy of each of the five methods here presented over as wide a range of values as those in this study.

III. Methodology

This chapter presents the methodology used to analyze the CEP approximation techniques. The variables used in the methods (mean, standard deviation, and correlation) are introduced, followed by a presentation of the mathematics of each of the approximation methods. Next the framework of the attempted regression analysis is discussed, followed by a description of the process by which data was generated for this study. Finally, the interactive computer program included in appendix E is described.

The Variables

Circular error probable can be expressed as a function of the standard deviation, mean, and correlation of the downrange and crossrange miss distances of a set of sample impacts. Correlation can be removed from sample data by performing a simple rotation of axes into a non-correlated coordinate system (see Appendix B for the this rotation technique and the formula for calculating correlation) (10:21). This study assumes one is using the standard deviations and means from uncorrelated samples.

The unbiased estimators of standard deviation and mean are used here. Equations 3.1 and 3.2 show the formulae used in calculating the down range and cross range sample means:

$$\bar{x} = \sum_{i=1}^{n} x_i / n \qquad (3.1)$$

$$\bar{y} = \sum_{i=1}^{n} y_i/n \qquad (3.2)$$

where x refers to cross range miss distance, y refers to down range miss distance, and n is the total number of sample impacts. Equations 3.3 and 3.4 are the formulae for calculating the sample standard deviations:

$$S_x = \sqrt{\sum_{i=1}^{n} (x_i^2 - n\bar{x}^2)/(n - 1)} \qquad (3.3)$$

$$S_y = \sqrt{\sum_{i=1}^{n} (y_i^2 - n\bar{y}^2)/(n - 1)} \qquad (3.4)$$

From these variables the parameters of bias and ellipticity are obtained.

$$\text{bias} = \sqrt{\bar{x}^2 + \bar{y}^2} \qquad (3.5)$$

$$\text{ellipticity} = S_x/S_y \qquad (3.6)$$

The relative errors between the approximation techniques and the "exact" approximation vary depending on the bias and ellipticity of a given set of sample impacts.

Grubbs-Patnaik/Chi-Square

The Grubbs-Patnaik/chi-square approximation method is based on the work of Frank E. Grubbs (7). This method uses the fact that the bias is a sum of noncentral chi-squares and solves for the radius that gives a probability of 0.5 (i.e. $P[(x-\bar{x})^2/S_x^2 + (y-\bar{y})^2/S_y^2 <= R^2] = 0.5$) (7:55).

This is the approximation:

$$CEP = \sqrt{kv/2m} \quad (3.7)$$

where

$$m = (S_x{}^2 + S_y{}^2 + \bar{x}^2 + \bar{y}^2) \quad (3.8)$$

$$v = 2(S_x{}^4 + 2\rho^2 S_x{}^2 S_y{}^2 + S_y{}^4) + 4(\bar{x}^2 S_x{}^2 + 2\bar{x}\bar{y}\rho S_x S_y + \bar{y}^2 S_y{}^2) \quad (3.9)$$

$$k = F^{-1}(0.5) \quad (3.10)$$

$$df = 2m^2/v \quad (3.11)$$

and F is the chi-square distribution function with df degrees of freedom (1:5). In this formula ρ is the correlation coefficient. If the rotation described in appendix B is performed then the terms with ρ in them can be ignored ($\rho = 0$).

In the computer programs used in this study the inverse chi-square function was estimated using table values found in the CRC Standard Math Tables (5:547). This table gives values of F^{-1} for integer values of df, but F^{-1} is a continuous function. To obtain values of F^{-1} for non-integer values of df a simple linear interpolation was used (Note: F^{-1} is not a linear function, but for the sake of simplicity the linear approximation was used for the interpolation. Comparing the results obtained using this simple linear interpolation and results of the Grubbs-Patnaik/chi-square

from other studies that used a more exact expression for F^{-1} showed that the was little or no loss of accuracy.).

<u>Grubbs-Patnaik/Wilson-Hilferty</u>

The Grubbs-Patnaik/Wilson-Hilferty approximation technique was developed as a modification of the Grubbs-Patnaik/chi-square method. The Grubbs-Patnaik/Wilson-Hilferty method transforms the chi-square to approximate normal variables. This method does not use the chi-square function described in the previous section. The expression for CEP used in the Grubbs-Patnaik/Wilson-Hilferty method is:

$$CEP = \sqrt{m\{1 - [v/(9m^2)]^3\}} \qquad (3.12)$$

where m and v are as defined in the Grubbs-Patnaik/chi-square method (equations 3.8 and 3.9) (1:6).

<u>Modified RAND-234</u>

The Modified RAND-234 method is a fit of a cubic polynomial to a table of CEP values. Pesapane and Irvine used regression analysis to "...derive a mathematical expression for circular error probable (CEP) which approximates probability tables contained in RAND Report R-234" (14:2). This is the Modified RAND-234 method:

$$CEP = CEP_{MPI}(1.0039 - 0.0528v + 0.4786v^2 - 0.0793v^3) \qquad (3.13)$$

where CEP_{MPI} is the CEP centered on the mean point of impact (this CEP must be translated to a CEP centered on the

target), S_S is the smaller of the two standard deviations, S_L is the larger, and:

$$CEP_{MPI} = 0.614S_S + 0.563S_L \tag{3.14}$$

$$S_S = \sqrt{\frac{S_x^2 + S_y^2 - \sqrt{(S_x^2 - S_y^2)^2 + 4\rho^2 S_x^2 S_y^2}}{2}} \tag{3.15}$$

$$S_L = \sqrt{\frac{S_x^2 + S_y^2 + \sqrt{(S_x^2 - S_y^2)^2 + 4\rho^2 S_x^2 S_y^2}}{2}} \tag{3.16}$$

$$v = b/CEP_{MPI} \tag{3.17}$$

$$b = \text{bias} = \sqrt{\bar{x}^2 + \bar{y}^2} \tag{3.18}$$

The Modified RAND-234 was developed under the boundary *condition that S_S/S_L is greater than 0.25 and that v is less* than or equal to 2.2 (1:2-3, 14:2-5). These boundary conditions exclude highly elliptical sample impact data sets ($S_S < S_L/4$) and those whose mean point of impact is over 2.2 times as far away from the target centroid as the CEP around the mean point of impact (CEP_{MPI}).

<u>Correlated Bivariate Normal</u>

The Correlated Bivariate Normal (CBN) method of approximating CEP is a method that estimates the bivariate normal function (in polar coordinates) by integrating with respect to r and summing over θ. Here is the development of the CBN approximation technique (equation 3.19 is another expression for the distribution function of the bivariate

normal):

$$P(r^*) = \frac{1}{2\pi S_x S_y \sqrt{1-\rho^2}} \int_0^{2\pi} \int_0^{r^*} \exp[-(ar^2+2br+c)]\ r\,dr\,d\theta \qquad (3.19)$$

where r^* ($P(r^*) = P(CEP) = 0.5$) is the radius for which one is solving, a and b are functions of the change in θ (equations 3.20 and 3.21), and c is a constant (equation 3.22). Finally, equations 3.23 and 3.24 give the formulae for the CBN approximation technique. In these equations N is the number of intervals the integral is divided into to give this approximation, $\Delta\theta = 2\pi/N$, a_i and b_i are as given in equations 3.20 and 3.21 for the current value of θ ($2\pi i/N$), and c is as given in equation 3.22. The more intervals, the more accurate the approximation (15:1-5).

$$a(\theta) = \frac{1}{2(1-\rho^2)}\left(\frac{\sin^2\theta}{S_x^{\,2}} - \frac{2\rho\sin\theta\cos\theta}{S_x S_y} + \frac{\cos^2\theta}{S_y^{\,2}}\right) \qquad (3.20)$$

$$b(\theta) = \frac{-1}{2(1-\rho^2)}\left(\frac{\bar{x}\sin\theta}{S_x^{\,2}} - \frac{\rho\bar{x}\cos\theta + \rho\bar{y}\sin\theta}{S_x S_y} + \frac{\bar{y}\cos\theta}{S_y^{\,2}}\right) \qquad (3.21)$$

$$c = \frac{1}{2(1-\rho^2)}\left(\frac{\bar{x}^2}{S_x^{\,2}} - \frac{2\rho\bar{x}\bar{y}}{S_x S_y} + \frac{\bar{y}^2}{S_y^{\,2}}\right) \qquad (3.22)$$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp[-x^2]\,dx \qquad (3.23)$$

$$
P(r^*) = \frac{\Delta C}{4\sqrt{\pi}\, S_x S_y \sqrt{1-\rho^2}} - \sum_{i=1}^{N} \left\{ \frac{1}{a_i} \exp\left(\frac{b_i^2 - a_i c}{a_i} \right) \right.
$$

$$
* \left[\frac{b_i}{\sqrt{a_i}} \left\{ \operatorname{erf}\left(\sqrt{a_i}\, r^* + \frac{b_i}{\sqrt{a_i}} \right) - \operatorname{erf}\left(\frac{b_i}{\sqrt{a_i}} \right) \right\} + \right.
$$

$$
\left.\left. \frac{1}{\sqrt{\pi}} \left\{ \exp\left(-a_i (r^*)^2 - 2 b_i r^* - \frac{b_i^2}{a_i} \right) - \exp - \left(\frac{b_i^2}{a_i} \right) \right\} \right] \right\} \quad (3.24)
$$

To obtain a CEP using the correlated bivariate normal method one must first make a guess at what the CEP actually is, solve equation 3.23 and compare the value to 0.5. If the value obtained is "close enough" to 0.5, one has a CEP. However if the probability given by the CBN is not "close enough" to 0.5 then one must iterate to a value that is closer to 0.5. To decide how close is "close enough" one must decide how accurate your estimate of CEP is to be. The secant method (described in Appendix C) is used to iterate for the value of the CEP in the CBN.

"Exact" Method

Like the correlated bivariate normal technique, the "exact" method estimates the bivariate normal function. In order to use this method the two coordinates of miss must be uncorrelated. This method uses Taylor series and infinite series expansion to estimate CEP. The "exact" method takes the bivariate normal function in polar form:

$$P(R) = \int_0^R \int_0^{2\pi} A \exp -\frac{1}{2}\left[\frac{(r\cos\theta - \bar{x})^2}{S_x^2} + \frac{(r\sin\theta - \bar{y})^2}{S_y^2}\right] r\,dr\,d\theta \quad (3.25)$$

where

$$A = \frac{1}{2\pi S_x S_y} \quad (3.26)$$

After series expansion this is the equation for the "exact" approximation technique:

$$P(R) = 0.5 = \frac{R^2}{2} D \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} x_k y_j \frac{(k+1)!(j+1)!}{(k+j+1)!} \quad (3.27)$$

where

$$D = \frac{1}{S_x S_y} \exp\left[-\left(\frac{\bar{x}^2}{2S_x^2} + \frac{\bar{y}^2}{2S_y^2}\right)\right] \quad (3.28)$$

$$x_k = \frac{(2k)!}{(k+1)!(k!)^2}\left(\frac{-R^2}{8S_x^2}\right)^k \sum_{i=0}^{k} \frac{k!}{(k-i)!(2i)!}\left(\frac{-2\bar{x}^2}{S_x^2}\right)^i \quad (3.29)$$

$$y_j = \frac{(2j)!}{(j+1)!(j!)^2}\left(\frac{-R^2}{8S_y^2}\right)^j \sum_{i=0}^{j} \frac{j!}{(j-i)!(2i)!}\left(\frac{-2\bar{y}^2}{S_y^2}\right)^i \quad (3.30)$$

As with the CBN to solve this one must first make a guess at the CEP, and then iterate to arrive at an answer that is accurate enough for one's purpose. The secant method was used in this case also (17:1-5).

Regression Analysis

This study also developed a correction factor using regression analysis for the approximation techniques to make the methods better estimators of CEP. A least squares multiple linear regression model was constructed with two independent variables. The regression analysis uses bias (b) and ellipticity (e) the independent variables and the relative error (RE) as the dependent variable. The regression model is of the form:

$$RE = \beta_o + \beta_1 b + \beta_2 e \tag{3.31}$$

where the β_i's are the regression coefficients for which one must solve.

In order to solve for the β_i's define the following vectors and matrices:

$$\mathbf{RE} = \begin{bmatrix} RE_1 \\ RE_2 \\ \cdot \\ \vdots \\ RE_n \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} 1 & b_1 & e_1 \\ 1 & b_2 & e_2 \\ \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots \\ 1 & b_n & e_n \end{bmatrix}$$

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_o \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

The least squares estimator of $\boldsymbol{\beta}$ is:

$$\boldsymbol{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{RE} \tag{3.32}$$

(9:392-395)

After solving for the estimated relative error (RE) one

simply divides the estimated CEP by one plus the relative error. Recalling equation 2.1:

$$\frac{CEP_{approx} - CEP_{exact}}{CEP_{exact}} = RE \qquad (2.1)$$

therefore,

$$CEP_{exact} = \frac{CEP_{approx}}{(1 + RE)} \qquad (3.34)$$

The results of this least squares multiple linear regression are discussed in chapter IV.

<u>The Computer Program</u>

The computer program included in this study is written in the Pascal programming language and it is compatible with personal computers running Turbo Pascal by Borland International (with some minor changes it will run on other versions of Pascal). The program operates interactively. It will compute CEPs for each approximation technique, only those requested by the user, or it will decide which technique is the most accurate (other than the "exact" method) for the given parameters and return a CEP for that technique only. The program decides which method is the "best" using a decision criteria based on the results of this study.

The computer runs to collect the data for this study were performed on the VAX computer at the Air Force

Institute of Technology, and on the SANYO MBC-550 personal computer (operating system: MS-DOS) owned by the author. This program is intended to run on a personal computer in the offices of persons concerned with calculating CEPs. The programs were also run on a Zenith Z-150 at the Air Force Institute of Technology. There were no differences in how the program ran on the SANYO and the Zenith. The timing analysis was based on time to run on the SANYO.

The program can easily be adapted for use with any version of the Pascal language. The listing of the computer program is included in appendix C.

Data Collection

To collect data for this study CEPs for all five of the approximation methods were calculated over a wide range of values of ellipticity and bias. Since the bias is the radial distance of the mean point of impact from the target centroid, bias was stepped out on the diagonal where $\bar{x} = \bar{y}$, whereas ellipticity was calculated by keeping S_y constant and varying S_x. The range of ellipticities considered was from 0.05 to 1.0, and the range of bias was from 50 to 1555. Overall, 440 data points were examined in this study. These 440 data points examined ellipticities from 1.0 to 0.05 in decrements of 0.05 and bias from 1555 to 70 ($\bar{x}$ and $\bar{y}$ from 1100 to 50 in decrements of 50), and a CEP was calculated for each of the five approximation methods for all combinations of ellipticity and bias.

Appendix D contains sample output from the data collected. The sample in Appendix D shows CEPs for ellipticity from 1.0 to 0.1 (in 0.1 decrements) and bias from 1555 to 141 ($\bar{x}$ and $\bar{y}$ from 1100 to 100 in decrements of 100).

Once all the data runs were completed, the analysis of the approximation methods could begin.

IV. Analysis and Discussion of Results

This chapter discusses the results of the analysis described in chapter III. Each of the approximation techniques is presented with a discussion of how accurate (compared to the "exact" method and the other approximations) the method is over the range of the bias/ellipticity parameter space. Also considered in the analysis of each approximation technique is the computer time each technique takes to give an answer for the CEP. Finally this chapter discusses the regression analysis developed to attempt to correct the approximations.

Grubbs-Patnaik/Chi-Square

The Grubbs-Patnaik/chi-square method gives relative errors (RE) that range from -0.0076 to 0.0684 (in several cases the absolute relative error was less than 0.0001). Recall equation 2.1:

$$RE = \frac{CEP_{approx} - CEP_{exact}}{CEP_{exact}} \qquad (2.1)$$

The Grubbs-Patnaik/chi-square approximation technique underestimates the CEP 28% of the time. Figure 4-1 shows the portion of the parameter space where the Grubbs-Patnaik/chi-square method is the best.

This approximation is the most accurate of the approximation methods. It is the most accurate method for 46% of the data points. Additionally the Grubbs-

Patnaik/chi-square method gives an answer in an average of 2 seconds.

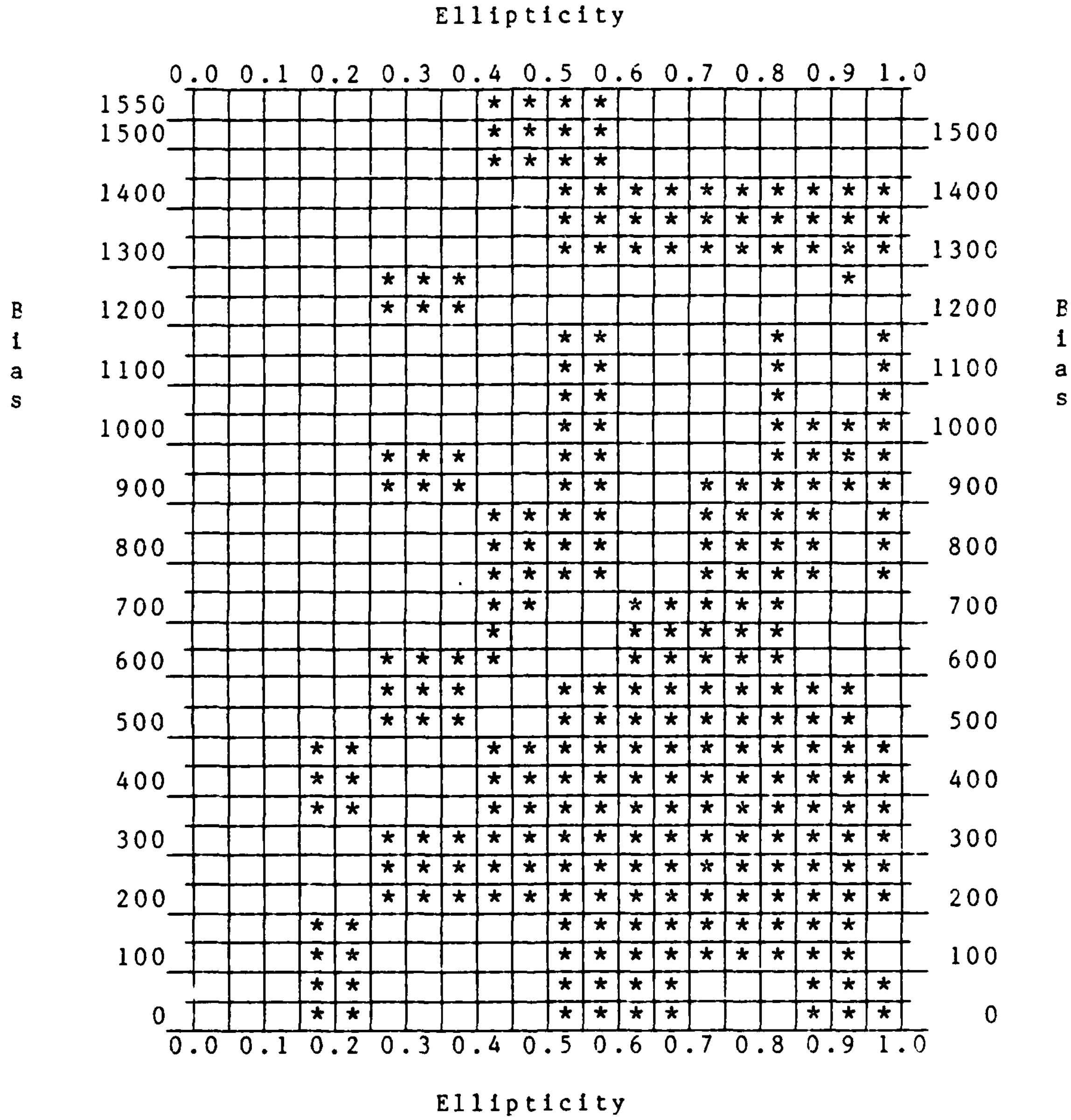

Figure 4-2
Grubbs-Patnaik/Chi-Square "Best"

Grubbs-Patnaik/Wilson-Hilferty

The Grubbs-Patnaik/Wilson-Hilferty method yields relative errors ranging from -0.054 to 0.0726 (the smallest

absolute relative error for this method was 0.0002). The percentage of underestimated CEPs given by this method is 10%. Figure 4-2 shows where this method is the most accurate method. The Grubbs-Patnaik/Wilson-Hilferty

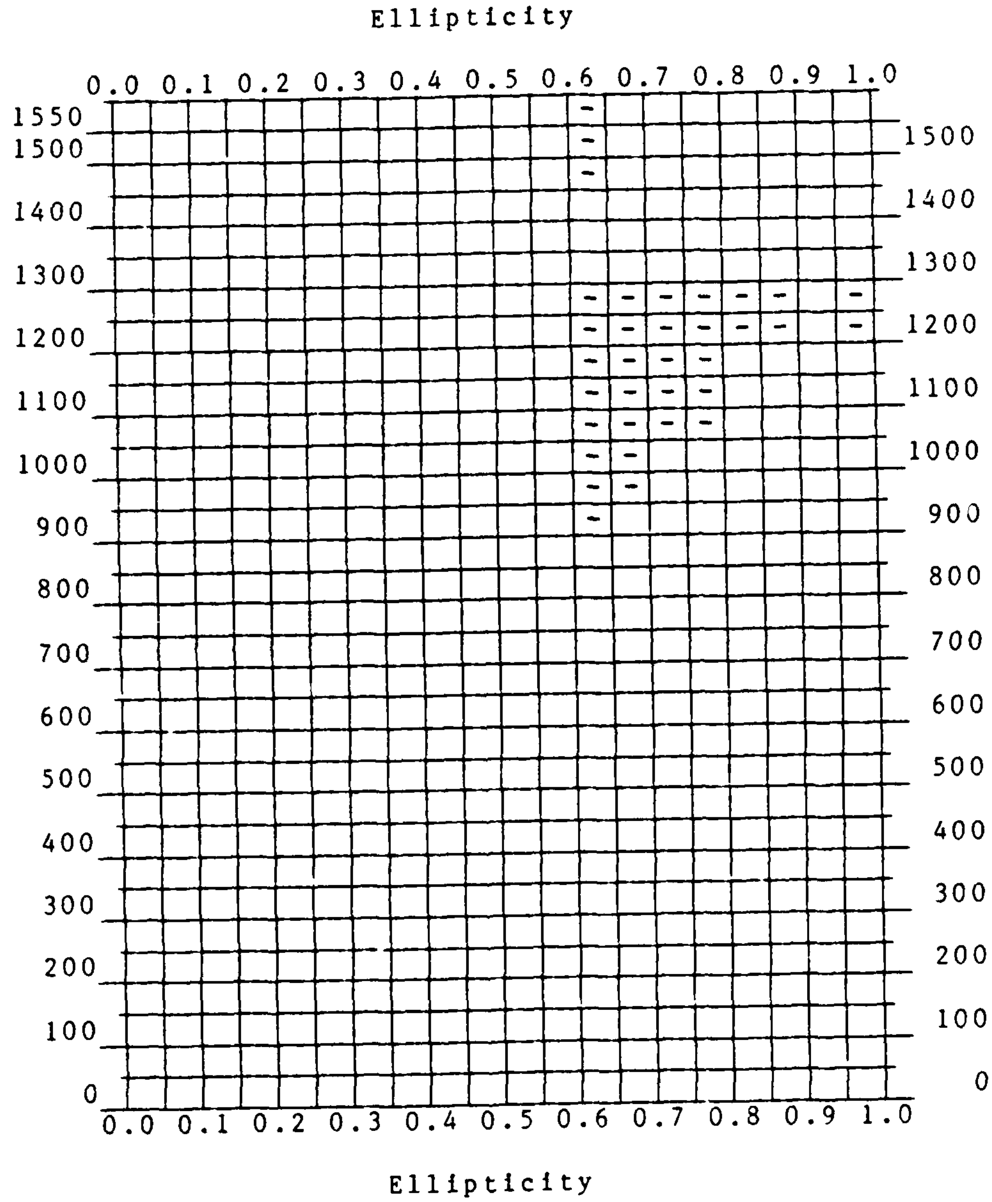

Figure 4-2
Grubbs-Patnaik/Wilson-Hilferty "Best"

approximation is the most accurate method for 5% of the data

points. Like the previous technique this method gives an answer in an average of 2 seconds.

Modified RAND-234

Values of -0.0125 to 0.0765 (0.0002 is the smallest

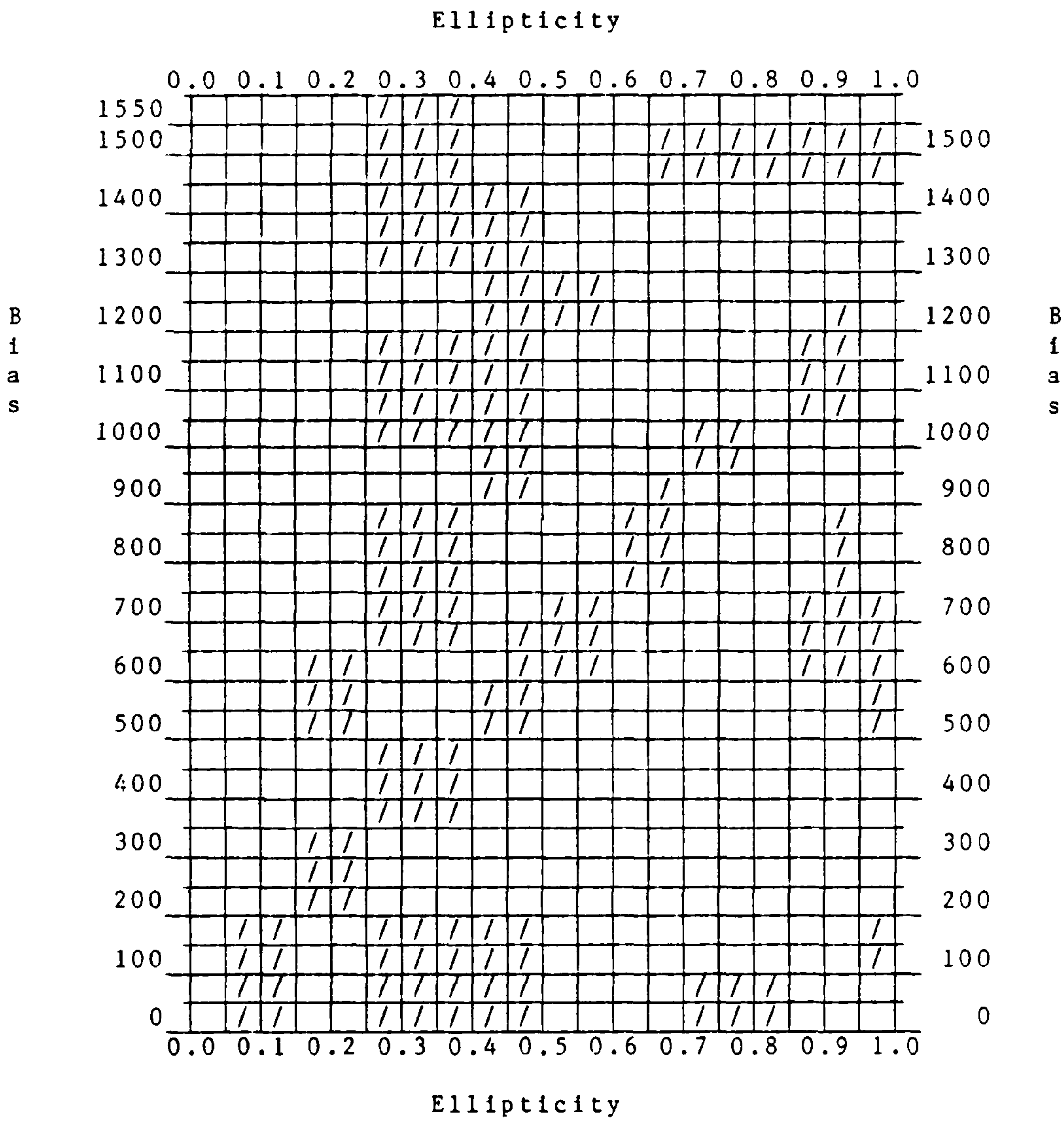

Figure 4-3

Modified RAND-234 "Best"

absolute relative error for this method) are the range of the relative errors given by the modified RAND-234 technique. It should also be noted that the modified RAND-234 method underestimates the CEP 60% of the time. Figure 4-3 shows where the modified RAND-234 method is most accurate in the bias/ellipticity parameter space. For 29% of the data points, the modified RAND-234 approximation gave the most accurate estimate of CEP when compared to the the other approximation methods. This approximation technique also returns estimates of CEP in an average of 2 seconds.

Astbury notes that the modified RAND-234 method was developed under the boundary condition that ellipticity, S_S/S_L, is greater than 0.25. However, the data for this study showed that the modified RAND-234 is also reliable for some values of ellipticity less than 0.25.

Correlated Bivariate Normal

The Correlated Bivariate Normal (CBN) approximation technique can be the most accurate estimate of CEP over the entire range of the parameter space if one has the time to wait for a result. This method takes anywhere from 10 minutes to several hours to produce a CEP that is accurate to within 0.001 absolute relative error. The CBN can return as accurate a value as desired. The more intervals the integral is broken into, the more accurate the approximation. At times the CBN takes longer than the "exact" method to give a CEP. However, the CBN always

converges to an estimate of CEP.

Here is an analysis of the time it takes the CBN to converge for three different numbers of intervals the integral is broken into:

40 intervals	-	10 minutes
100 intervals	-	1 hour
400 intervals	-	16 hours

As one can see, the amount of time it takes for the CBN to converge increases exponentially. Moreover, to achieve accuracies to within 1% it is often necessary to increase the number of intervals in the CBN.

"Exact"

The "exact" method is the benchmark against which all the other approximation techniques were measured. This method takes anywhere from 2 minutes to 2 hours to give an answer for CEP when it converges. The "exact" method does not always converge to a CEP. For some highly elliptical distributions with large biases where the target centroid is not within 2 standard deviations of the mean point of impact, the "exact" method diverges. Figure 4-4 shows the regimes of the bias/ellipticity parameter space where the "exact" method does not converge.

In the other regimes of the parameter space, the "exact" will converge if one has the time necessary for it to do so. If quick results are not necessary, then the

"exact" method is the preferred method to calculate CEP.

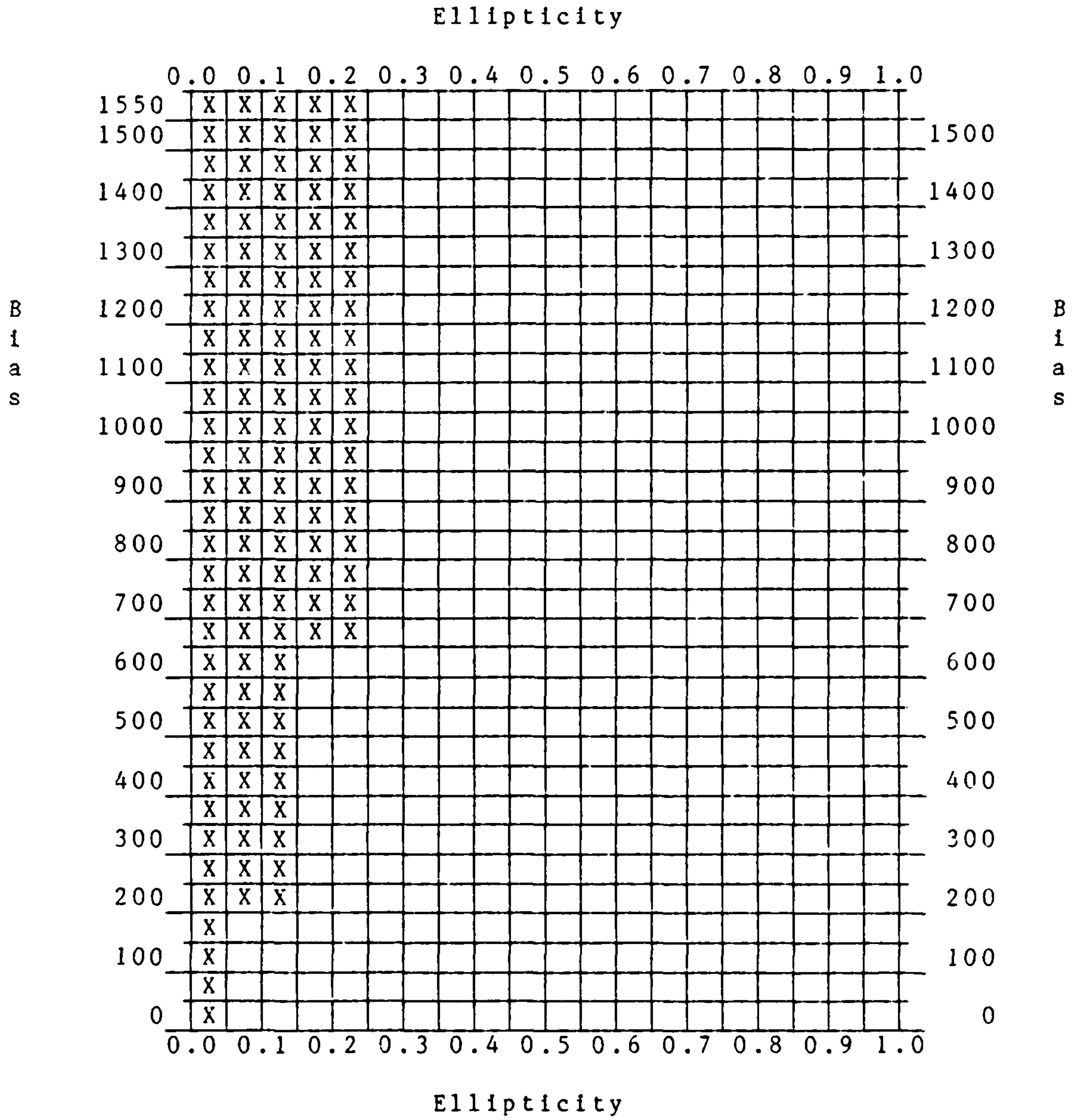

Figure 4-4

"Exact" Does Not Converge

Ellipticity

0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

Bias

1550	X	X	X	X	X	/	/	/	*	*	*	*	-	/	/	/	/	/	/	/	
1500	X	X	X	X	X	/	/	/	*	*	*	*	-	/	/	/	/	/	/	/	1500
	X	X	X	X	X	/	/	/	*	*	*	*	-	/	/	/	/	/	/	/	
1400	X	X	X	X	X	/	/	/	/	/	*	*	*	*	*	*	*	*	*	*	1400
	X	X	X	X	X	/	/	/	/	/	*	*	*	*	*	*	*	*	*	*	
1300	X	X	X	X	X	/	/	/	/	/	*	*	*	*	*	*	*	*	*	*	1300
	X	X	X	X	X	*	*	*	/	/	/	/	-	-	-	-	-	-	*	-	
1200	X	X	X	X	X	*	*	*	/	/	/	/	-	-	-	-	-	-	/	-	1200
	X	X	X	X	X	/	/	/	/	/	*	*	-	-	-	-	*	/	/	*	
1100	X	X	X	X	X	/	/	/	/	/	*	*	-	-	-	-	*	/	/	*	1100
	X	X	X	X	X	/	/	/	/	/	*	*	-	-	-	-	*	/	/	*	
1000	X	X	X	X	X	/	/	/	/	/	*	*	-	-	/	/	*	*	*	*	1000
	X	X	X	X	X	*	*	*	/	/	*	*	-	-	/	/	*	*	*	*	
900	X	X	X	X	X	*	*	*	/	/	*	*	-	/	*	*	*	*	*	*	900
	X	X	X	X	X	/	/	/	*	*	*	*	/	/	*	*	*	*	/	*	
800	X	X	X	X	X	/	/	/	*	*	*	*	/	/	*	*	*	*	/	*	800
	X	X	X	X	X	/	/	/	*	*	*	*	/	/	*	*	*	*	/	*	
700	X	X	X	X	X	/	/	/	*	*	/	/	*	*	*	*	*	/	/	/	700
	X	X	X	X	X	/	/	/	*	/	/	/	*	*	*	*	*	/	/	/	
600	X	X	X	/	/	*	*	*	*	/	/	/	*	*	*	*	*	/	/	/	600
	X	X	X	/	/	*	*	*	/	/	*	*	*	*	*	*	*	*	*	/	
500	X	X	X	/	/	*	*	*	/	/	*	*	*	*	*	*	*	*	*	/	500
	X	X	X	*	*	/	/	/	*	*	*	*	*	*	*	*	*	*	*	*	
400	X	X	X	*	*	/	/	/	*	*	*	*	*	*	*	*	*	*	*	*	400
	X	X	X	*	*	/	/	/	*	*	*	*	*	*	*	*	*	*	*	*	
300	X	X	X	/	/	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	300
	X	X	X	/	/	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	
200	X	X	X	/	/	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	200
	X	/	/	*	*	/	/	/	/	/	*	*	*	*	*	*	*	*	*	/	
100	X	/	/	*	*	/	/	/	/	/	*	*	*	*	*	*	*	*	*	/	100
	X	/	/	*	*	/	/	/	/	/	*	*	*	*	/	/	/	*	*	*	
0	X	/	/	*	*	/	/	/	/	/	*	*	*	*	/	/	/	*	*	*	0

0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

Ellipticity

Figure 4-5

"Best" Method Across the Bias/Ellipticity Parameter Space

/ = RAND-234
* = Grubbs-Patnaik/chi-square
- = Grubbs-Patnaik/Wilson-Hilferty
X = Area where the "exact" does not converge

For some values of ellipticity and bias, the "exact" method converges so quickly that it may be unnecessary to use a less accurate approximation. The folowing is a list of how long it takes the "exact" method to give a result for various values of the parameter space:

5 - 15 minutes

for $S_x > \bar{x}$ and $S_y > \bar{y}$

20 - 30 minutes

for $0.8 < S_x/S_y < 1.0$ and $S_x < \bar{x}$ and $S_y < \bar{y}$

30 - 90 minutes

for $0.5 < S_x/S_y < 0.8$ and $S_x < \bar{x}$ and $S_y < \bar{y}$

> 90 minutes

for all other values of ellipticity and bias.

The *notable* exceptions to the above analysis comes when S_x equals $\bar{x}$ or S_y equals $\bar{y}$. For these values the "exact" approximation method gives an answer for CEP in less than 2 minutes. Clearly there are times when it is feasible to use the "exact" method for calculating CEP. Again, use of the "exact" method would depend on the amount of computational effort one is willing to expend to achieve an answer.

Figure 4-5 maps the entire bias/ellipticity parameter space examined in this study and where each of the methods give the most accurate estimates of CEP. It should be noted that in those regimes where the "exact" method will not converge, one can use the CBN to achieve a highly accurate estimate of CEP (if the time is available).

Regression Analysis

As part of this study a multiple linear regression was performed on three of the approximation techniques (Grubbs-Patnaik/chi-square, Grubbs-Patnaik/Wilson-Hilferty, and the modified RAND-234). This regression analysis was performed in the hope of providing a correction factor for the approximations so they would more closely estimate CEP. This regression analysis was based on the relative errors.

Here are the three regression equations developed using the regression procedure described in Chapter III (RE stands for relative error):

Grubbs-Patnaik/chi-square

$$RE = -0.000256 + 0.000392(\text{Ellip}) + 0.00000172(\text{bias}) \quad (4.1)$$

Grubbs-Patnaik/Wilson-Hilferty

$$RE = -0.000472 + 0.000693(\text{Ellip}) + 0.00000395(\text{bias}) \quad (4.2)$$

Modified RAND-234

$$RE = -0.000403 + 0.000498(\text{Ellip}) + 0.00000316(\text{bias}) \quad (4.3)$$

Once an estimate of RE is determined, recall equation 3.30:

$$CEP_{exact} = \frac{CEP_{approx}}{(1 + RE)} \quad (3.30)$$

is used to come up with a better approximation to the "exact".

These regression equations give good results for CEP

estimates that are greater than the "exact" CEP. In other words, they do improve the estimate of the CEP. However they do not capture the pattern that causes the approximation methods to underestimate the CEP. Using the Grubbs-Patnaik/chi-square method here is an example of the results of the regression:

ellipticity = 0.5	bias = 282.84
CEP(approx.) = 907.75	CEP("exact") = 906.50
RE(actual) = 0.0014	RE(regression) = 0.0004

Using the regression technique described one would end up with a CEP of 907.36, which has a relative error of 0.0009. That is some improvement from the original. However, when the method underestimates the CEP this is a typical result:

ellipticity = 0.7	bias = 848.53
CEP(approx.) = 1252.7	CEP("exact") = 1254.65
RE(actual) = -0.0016	RE(regression) = 0.0015

After using the described regression technique the CEP is 1250.8. This is a number further from the "exact" CEP than the original approximation.

Similar results were obtained for each of the approximation techniques when a suitable correction factor was desired and the approximation had underestimated the CEP. On the average, the regression equation for the Grubbs-Patnaik/chi-square method improves the relative error between the approximation and the "exact" by 0.0005. The

regression for the Grubbs-Patnaik/Wilson-Hilferty results in an improvement of 0.0008 on the average. The modified RAND-234 regression gives an average improvement in relative error of 0.0003. The fact remains that the regressions do not reliably predict the times that the approximations underestimate the CEP. It seems as if the relative errors of the approximation techniques are not well behaved enough to be captured in a simple equation. Appendix D table D-3 contains a listing of the relative errors obtained for some example values of ellipticity and bias.

The author also tried several variations for the regression independent variables. Squaring the ellipticity and bias was attempted as well as taking the logarithms of the variables. All attempts at providing a reliable correction factor for the algebraic approximation methods were unsuccessful.

The scope of this study was by no means completely exhaustive. However, the author is able to make some conclusions and recommendations based on the analysis of the results of this study.

V. Conclusions and Recommendations

Conclusions

Circular error probable (CEP) is an important measure of a weapon system's accuracy. As such it is essential to be able to closely approximate this number. This study has examined five different methods of estimating CEP and from this examination the author has reached several conclusions regarding CEP calculations.

First and foremost among the author's conclusions is that among the approximation methods studied, CEP is not "well behaved." What is meant by that, is that there is no easily recognizable pattern among the approximations as to which is most accurate over a given range of the parameter space. This is evidenced by the failure of the regression analysis that was attempted in this study. Had the differences between the algebraic approximations and the "exact" method been somewhat "well behaved" the regression would have provided a much more reliable correction factor. However, this was not the case.

Another observation is that the algebraic approximations examined (Grubbs-Patnaik/chi-square, Grubbs-Patnaik/Wilson-Hilferty, and modified RAND-234) are approximately 99% accurate for most values of ellipticity and bias when compared to the "exact" method. Notable exceptions to this are where $\bar{x}$ equals S_x or $\bar{y}$ equals S_y. The resulting values of ellipticity and bias cause the

accuracies of the algebraic approximations to fall to between 90% and 98%. However, for these values the "exact" method produces an answer in seconds. Highly elliptical distributions cause the greatest errors in all methods (if ellipticity is less than 0.3, accuracies drop to between 95% and 99%).

The correlated bivariate normal (CBN) method is the most consistent method of all. However, for this approximation method to deliver accuracies better than 99%, the CBN often requires hours to produce an estimate of the CEP. The only time that the CBN would be the preferred method is when extremely accurate results (better than 99%) are needed for highly elliptical cases where the "exact" method fails to converge. For these results to be accurate to better than 0.1%, one must increase the number of intervals for the integral approximation to over 100, and this required several hours on the author's SANYO MBC-550 personal computer.

The "exact" method, in some regimes, is very practical to use. If S_x is greater than or equal to $\bar{x}$ and S_y is greater than or equal to $\bar{y}$ the "exact" method converges to the CEP very fast (less than 8 minutes). The "exact" method has one major drawback and that is that it requires a computer to run the iterations (this is also true of the CBN). The algebraic approximation techniques involve only a few equations that are solved once. These could conceivably be programmed into a hand-held programmable calculator. The

"exact" method requires series expansion until inner terms approach zero and then an iterative scheme to achieve a desired accuracy in the calculation. In the areas where the "exact" method does not converge, one must rely on the other approximations for estimates of CEP.

Recommendations

Based on the difficulties this author found in programming these approximations, Pascal is probably not the best language for these calculations. Pascal has some limitations on the size of the real numbers it will accept. Turbo Pascal assigns real numbers to a 6 byte word size with no provision for double precision reals. This limits the range of real numbers to 1E-38 to 1E+38. Many of the calculations required in the approximation techniques produce numbers out of the range of those acceptable in Pascal. The author was reduced to using logarithms and other manipulations to bypass these temporary large numbers. This resulted in some inefficiencies in the programs which could lead to longer processing time. The author also found this same problem with the version of Pascal that is available on the VAX computer at the Air Force Institute of Technology. The author is aware of the existence of other versions of Pascal that can handle double precision real numbers. This version of Pascal was not available for use in this study.

There are other computer languages that would perhaps

lend themselves more readily to the great number of calculations required in calculating CEP. It may be of use to those who calculate CEPs to have the programs furnished with this study translated into one of these other languages.

In addition to trying other computer languages this author is not convinced that there is no way to capture the essence of the errors in the algebraic approximations in order to produce a correction factor for these methods. This portion of this study warrants further efforts.

The approximation techniques examined in this study were all based on the assumption that the distribution of impacts follows the bivariate normal distribution. It would be useful to have robust techniques developed that could provide good estimates of CEP (given small samples) that are independent of the distribution of the impacts. This would obviate the need to make any assumptions about the distribution of the impacts.

Finally, this author recommends that if one needs highly accurate estimates of CEP and sufficient time is available, the "exact" method should be used for all values where the "exact" converges to a CEP. In those cases where the "exact" does not converge, the CBN with at least 100 intervals should be used.

Otherwise one will be forced to use one of the algebraic approximations. If one is to chose a single

approximation technique to use, the Grubbs-Patnaik/chi-square is the most accurate for the greatest range of values of ellipticity and bias. The analysis presented in this study will aid the user in determining which approximation is the "best" for one's purposes.

Appendix A

Circular Error Probable Tutorial

Introduction

Whether one is dealing with guided bombs, unguided bombs, bullets, missiles, rockets, lasers, etc., for a weapons systems planner to effectively plan the employment of a weapon there must be some measure of that weapon's accuracy. Circular error probable (CEP) is one measure of accuracy that is frequently used. CEP is defined as:

> ... the radius, centered on the target, within which the probability of impact is 0.5.

To determine a weapon's CEP one must test the weapon to get a collection of sample impacts. This collection of sample impacts may be large (greater than 30) or it may be quite small (take the case of a multi-million dollar missile - one wouldn't launch 30 or more of them just to test their accuracy). In the case of a large sample size, non-parametric techniques (i.e. those that make no assumptions about the underlying statistical distribution) are adequate for estimating the CEP. But non-parametric techniques are subject to large errors when dealing with small sample sizes. In the case of small sample sizes, one must resort to parametric methods for approximating CEP. These parametric techniques assume a particular statistical distribution and estimate the various parameters of that

distribution (i.e. mean, standard deviation, correlation, etc.).

This tutorial gives a brief introduction to these various techniques for estimating CEP starting with non-parametric techniques followed by a discussion of parametric techniques. Finally, results of CEP calculations for 10 sample impacts of a hypothetical ICBM warhead using each of the approximation techniques is presented.

Non-Parametric Approximation

A good non-parametric approximation for CEP is the sample median. To determine the sample median, one must rank order the sample miss distances (that is the straight line distance from the actual impact point to the target). The sample median ($\tilde{X}$) is the middle statistic (or if there are an even number of sample points, the median is the average of the two middle order statistics of the sample points).

$$\tilde{X} = X_{((n+1)/2)} \qquad \text{if } n \text{ is odd,} \qquad (A.1)$$

$$\tilde{X} = \{X_{(n/2)} + X_{((n+1)/2)}\}/2 \qquad \text{if } n \text{ is even.} \qquad (A.2)$$

The sample median is a good statistic to use (if one has a large sample) because it requires no assumptions about the underlying distribution of the impacts.

Parametric Approximations

Parametric approximations assume that there is an underlying distribution to the impacts. This report

presents parametric approximation techniques that assume the distribution of impacts is bivariate normal about some mean point of impact. There are several parametric approximations available for estimating CEP. Most of them are based on the means and standard deviations of the downrange and crossrange miss distances of the sample impacts. Figure A-1 is an example of how to determine the downrange and crossrange miss distances.

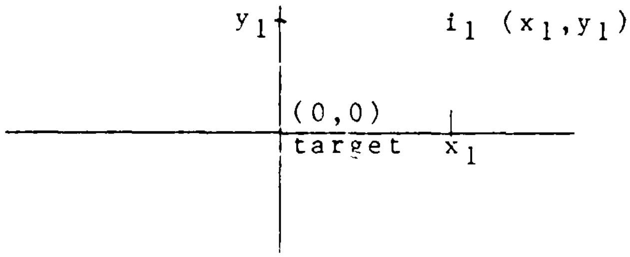

Figure A-1 Sample Impact i_1

Set up a coordinate system with the target as the origin. Define x_1 as the crossrange miss distance and y_1 as the downrange miss distance for sample impact #1 (i_1). Then take the total number of sample impacts (n) and calculate the crossrange and downrange miss distances for each of them.

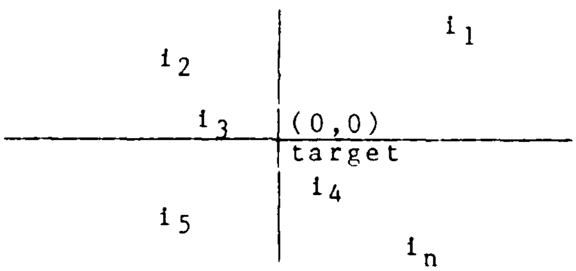

Figure A-2 n - Sample Impacts

Next calculate the mean and standard deviation for the sample crossrange and downrange miss distances.

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad (A.3) \qquad\qquad S_x = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{(n - 1)}} \qquad (A.4)$$

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \qquad (A.5) \qquad\qquad S_y = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{(n - 1)}} \qquad (A.6)$$

The means will give you a mean point of impact (MPI)

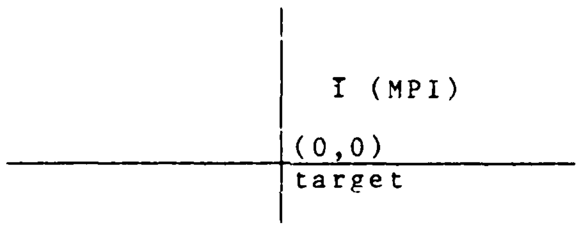

Figure A-3 Mean Point of Impact

If the MPI and the target coincide, then the CEP is relatively easy to calculate. However, this is not always the case. For cases where the MPI and the target do not coincide then the CEP_{MPI} must be translated to a CEP centered on the target.

<u>Calculating</u> <u>CEP.</u> The probability distribution function of the crossrange miss distance is:

$$f(x) = \frac{\exp[-0.5(x/S_x)^2]}{\sqrt{2\pi}\; S_x} \qquad -\infty < x < \infty \qquad (A.7)$$

and the probability distribution function of the downrange miss distances is:

$$f(y) = \frac{\exp[-0.5(y/S_y)^2]}{\sqrt{2\pi}\ S_y} \qquad -\infty < y < \infty \qquad (A.8)$$

If it is assumed that $S = S_x = S_y$, and that the crossrange and downrange miss distances are independent, then the joint distribution of the miss distances is:

$$f(x,y) = \frac{\exp\{-0.5[(x^2 + y^2)/S^2]\}}{2\pi\ S^2} \qquad (A.9)$$

Changing to polar coordinates, the joint distribution is:

$$f(r,\theta) = \frac{r\ \exp[-0.5(r/S)^2]}{2\pi\ S^2} \qquad \begin{matrix} 0 < r < \infty \\ 0 < \theta < 2\pi \end{matrix} \qquad (A.10)$$

To solve for a particular radius r^* such that The probability that r is less than r^* equals 0.5 (r^* = CEP) one must integrate equation A.10 with respect to θ (0 to 2π) and with respect to r (0 to r^*).

This leaves :

$$P(r < r^*) = 0.5 = 1 - \exp[-0.5(r^*/s)^2] \qquad (A.11)$$

and

$$r^* = 1.1774\ S = CEP \qquad (A.12)$$

In the case of $\bar{x} = \bar{y} = 0$ and $S_x \neq S_y$ the formula for the joint distribution is:

$$f(r,\theta) = \frac{1}{2\pi S_x S_y} \exp\left\{-\frac{1}{2}\left[\left(\frac{r\ \cos\theta}{S_x}\right)^2 + \left(\frac{r\ \sin\theta}{S_y}\right)^2\right]\right\} \qquad (A.13)$$

If $0.25 < S_x/S_y < 1$ this expression (after integration) reduces to:

$$r^* = 0.614S_x + 0.563S_y = CEP \qquad (A.14)$$

(5:5). For other values of S_x, S_y and for samples not centered on the target, one is left to use other approximations to determine the CEP.

Popular Approximation Methods. There are several approximation methods available for estimating CEP in elliptic, biased samples (i.e. $S_x \neq S_y$ and $\bar{x} \neq \bar{y} \neq 0$). Presented here are four of the approximation methods available.

Grubbs-Patnaik/Chi-Square. This approximation method is based on the work of Frank E. Grubbs (2). This method uses the fact that the bias is a sum of noncentral chi-squares and solves for the radius that gives a probability of 0.5 (i.e. $P[(x-\bar{x})^2/S_x^2 + (y-\bar{y})^2/S_y^2 <= R^2] = 0.5$) (2:55). This is the approximation:

$$CEP = \sqrt{kv/2m} \qquad (A.15)$$

where

$$m = (S_x^2 + S_y^2 + \bar{x}^2 + \bar{y}^2) \qquad (A.16)$$

$$v = 2(S_x^4 + 2\rho^2 S_x^2 S_y^2 + S_y^4) + 4(\bar{x}^2 S_x^2 + 2\bar{x}\bar{y}\rho S_x S_y + \bar{y}^2 S_y^2) \qquad (A.17)$$

$$k = F^{-1}(0.5) \qquad (A.18)$$

$$n = 2m^2/v \qquad (A.19)$$

and F is the chi-square distribution function with n degrees of freedom (1:5). In this formula ρ is the correlation coefficient. If the correlation coefficient is non-zero, one can perform a rotation of axes into a coordinate system where the two coordinates of miss distance are uncorrelated (3:20-21). Attachment 1 to this tutorial describes how to find the correlation coefficient and how to perform this rotation to eliminate the correlation (this makes $\rho = 0$).

Grubbs-Patnaik/Wilson-Hilferty. This approximation technique was developed as a modification of the Grubbs-Patnaik/chi-square method. The Wilson-Hilferty method transforms the chi-square to approximate normal variables. This method does not use the chi-square function described in the previous section. Here is the Grubbs-Patnaik/Wilson-Hilferty method:

$$CEP = \sqrt{m\{1 - [v/(9m^2)]^3\}} \qquad (A.20)$$

where m and v are as defined in the Grubbs-Patnaik/chi-square method (equations A.16 and A.17) (D:6).

Modified RAND-234. This method is a fit of a cubic polynomial to a table of CEP values. Pesapane and Irvine used regression analysis to "...derive a mathematical expression for circular error probable (CEP) which approximates probability tables contained in RAND Report R-234" (N:2). This is the Modified RAND-234 method:

$$CEP = CEP_{MPI}(1.0039 - 0.0528v + 0.4786v^2 - 0.0793v^3) \qquad (A.21)$$

where

$$CEP_{MPI} = 0.614S_S + 0.563S_L \qquad (A.22)$$

$$S_S = \sqrt{\frac{S_x^2 + S_y^2 - \sqrt{(S_x^2 - S_y^2)^2 + 4\rho^2 S_x^2 S_y^2}}{2}} \qquad (A.23)$$

$$S_L = \sqrt{\frac{S_x^2 + S_y^2 + \sqrt{(S_x^2 - S_y^2)^2 + 4\rho^2 S_x^2 S_y^2}}{2}} \qquad (A.24)$$

$$v = b/CEP_{MPI} \qquad (A.25)$$

$$b = \sqrt{\bar{x}^2 + \bar{y}^2} \qquad (A.26)$$

The Modified RAND-234 was developed under the boundary condition that S_S/S_L is greater than 0.25 and that v is less than or equal to 2.2 (1:2-3, 4:2-5). If one has a correlated sample as described in the section on the Grubbs-Patnaik/chi-square method, the same rotation can be used to remove the correlation.

"Exact" Method. The "exact" method, despite its name, is also an approximation technique. It is recognized as the most accurate approximation method available. This method is a numerical integration technique used to estimate the integral of the bivariate normal distribution. This method uses Taylor series and infinite series expansion to estimate CEP. The "exact" method is computationally rigorous and requires a computer for its calculations. Since it is so computationally rigorous it is often not feasible to use the

"exact" method and that is why one must resort to using less accurate approximations. However, the "exact" method is useful for determining which approximation technique is the most accurate.

The "exact" method takes the bivariate normal function in polar form:

$$P(R) = \int_0^R \int_0^{2\pi} A \exp\left[-\frac{1}{2}\left(\frac{(r\cos\theta - \bar{x})^2}{S_x^2} + \frac{(r\sin\theta - \bar{y})^2}{S_y^2}\right)\right] r\,dr\,d\theta \quad (A.27)$$

where

$$A = \frac{1}{2\pi S_x S_y} \quad (A.28)$$

After series expansion this is the equation for the "exact" approximation technique:

$$P(R) = 0.5 = \frac{R^2}{2} D \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} x_k y_j \frac{(k+1)!(j+1)!}{(k+j+1)!} \quad (A.29)$$

where

$$D = \frac{1}{S_x S_y} \exp\left\{-\left[\frac{\bar{x}^2}{2S_x^2} + \frac{\bar{y}^2}{2S_y^2}\right]\right\} \quad (A.30)$$

$$x_k = \frac{(2k)!}{(k+1)!(k!)^2} \left[\frac{-R^2}{8S_x^2}\right]^k \sum_{l=0}^{k} \frac{k!}{(k-l)!(2l)!} \left[\frac{-2\bar{x}^2}{S_x^2}\right]^l \quad (A.31)$$

$$y_j = \frac{(2j)!}{(j+1)!(j!)^2} \left[\frac{-R^2}{8S_y^2}\right]^j \sum_{i=0}^{j} \frac{j!}{(j-i)!(2i)!} \left[\frac{-2\bar{y}^2}{S_y^2}\right]^i \quad (A.32)$$

(6:1-5)

To compute this approximation, one must first make a guess at the CEP (it is recommended to use one of the fast algebraic approximation methods tc arrive at your guess), and then iterate to arrive at an answer that is accurate enough for one's purpose. This method sometimes requires a great deal of computer time to arrive at an answer. But it is the most accurate estimator of CEP.

Sample Calculations

Now that one has the approximation methods, the next step is to actually calculate CEPs. The following is an example of how one would do this for each of the presented approximation techniques.

Table A-1 contains sample crossrange and downrange miss distances for 10 sample impacts of a theoretical new warhead (Note: this table uses feet as an illustration, the methods, however, are insensitive to units as long as one is consistent with the units used). To calculate the CEP using

Table A-1
Sample Impacts for a Theoretical ICBM Warhead

Impact	Crossrange Miss Distance (x)	Downrange Miss Distance (x)	Radial Miss Distance
1	400 ft	569 ft	695.5 ft
2	324	429	545.6
3	-116	125	170.5
4	50	-214	219.8
5	-68	-126	143.2
6	257	302	396.6
7	76	-156	173.5
8	96	158	184.9
9	-30	53	60.9
10	155	204	256.2

the non-parametric technique described (the sample median) one must first rank order the sample radial miss distances as in table A-2. The radial miss distance (also called the bias) is defined as the square root of the sum of the squares of the crossrange and downrange miss distances. The numbers in parentheses in table A-2 indicate that these are order statistics.

Table A-2
Sample Order Statistics for Radial Miss Distance

(Ordered Impact)	Radial Miss Distance
(1)	60.9
(2)	143.2
(3)	170.5
(4)	173.5
(5)	184.9
(6)	219.8
(7)	256.2
(8)	396.6
(9)	545.6
(10)	695.5

Using equation A.2 one calculates the sample median, 202.35, and this serves as a non-parametric estimate of the CEP.

Next, the parametric estimates of CEP are calculated. To do this one must calculate the sample crossrange and downrange means and standard deviations and the correlation coefficient. Here are those values:

Crossrange Mean: $\bar{x} = 120.4$

Crossrange Standard Deviation: $S_x = 165.5$

Downrange Mean: $\bar{y} = 134.4$

Downrange Standard Deviation: $S_x = 255.9$

Correlation Coefficient: $\rho = 0.78$

Since the correlation coefficient is non-zero, one must perform the rotation discussed in attachment 1. After this is performed, these are the values to be substituted for $\bar{x}$, S_x, $\bar{y}$, and S_y:

$$\bar{x}' = 37.1$$

$$\bar{y}' = 281.8$$

$$S_x' = 91.1$$

$$S_y' = 290.7$$

$$\rho = 0$$

Using the above values the results of the approximations are:

Grubbs-Patnaik/chi-square:	CEP = 324.34
Grubbs-Patnaik/Wilson-Hilferty:	CEP = 325.18
Modified RAND-234:	CEP = 343.76
"Exact":	CEP = 315.75

Compare these values to the non-parametric estimate, 202.35, and one can see how inaccurate the non-parametric estimate is, given a small sample size. The calculations required for the "exact" method can often require a great deal of computer time (anywhere from 2 minutes to 2 hours, and sometimes it does not converge to an answer at all), so it is useful to know which other approximation methods are most accurate. In the above case the Grubbs-Patnaik/chi-square approximation technique is the best (closest to the "exact")

and, on the average, the Grubbs-Patnaik/chi-square is the most accurate approximation method.

Ellipticity

0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

Bias																					Bias
1550	X	X	X	X	X	/	/	/	*	*	*	*	-	/	/	/	/	/	/	/	
1500	X	X	X	X	X	/	/	/	*	*	*	*	-	/	/	/	/	/	/	/	1500
	X	X	X	X	X	/	/	/	*	*	*	*	-	/	/	/	/	/	/	/	
1400	X	X	X	X	X	/	/	/	/	/	*	*	*	*	*	*	*	*	*	*	1400
	X	X	X	X	X	/	/	/	/	/	*	*	*	*	*	*	*	*	*	*	
1300	X	X	X	X	X	/	/	/	/	/	*	*	*	*	*	*	*	*	*	*	1300
	X	X	X	X	X	*	*	*	/	/	/	/	-	-	-	-	-	-	*	-	
1200	X	X	X	X	X	*	*	*	/	/	/	/	-	-	-	-	-	-	/	-	1200
	X	X	X	X	X	/	/	/	/	/	*	*	-	-	-	-	*	/	/	*	
1100	X	X	X	X	X	/	/	/	/	/	*	*	-	-	-	-	*	/	/	*	1100
	X	X	X	X	X	/	/	/	/	/	*	*	-	-	-	-	*	/	/	*	
1000	X	X	X	X	X	/	/	/	/	/	*	*	-	-	/	/	*	*	*	*	1000
	X	v	X	X	X	*	*	*	/	/	*	*	-	-	/	/	*	*	*	*	
900	X	X	X	X	X	*	*	*	/	/	*	*	-	/	*	*	*	*	*	*	900
	X	X	X	X	X	/	/	/	*	*	*	*	/	/	*	*	*	*	/	*	
800	X	X	X	X	X	/	/	/	*	*	*	*	/	/	*	*	*	*	/	*	800
	X	X	X	X	X	/	/	/	*	*	*	*	/	/	*	*	*	*	/	*	
700	X	X	X	X	X	/	/	/	*	*	/	/	*	*	*	*	*	/	/	/	700
	X	X	X	X	Y	/	/	/	*	/	/	/	*	*	*	*	*	/	/	/	
600	X	X	X	/	/	*	*	*	*	/	/	/	*	*	*	*	*	/	/	/	600
	X	X	X	/	/	*	*	*	/	/	*	*	*	*	*	*	*	*	*	/	
500	X	X	X	/	/	*	*	*	/	/	*	*	*	*	*	*	*	*	*	/	500
	X	X	X	*	*	/	/	/	*	*	*	*	*	*	*	*	*	*	*	*	
400	X	X	X	*	*	/	/	/	*	*	*	*	*	*	*	*	*	*	*	*	400
	X	X	X	*	*	/	/	/	*	*	*	*	*	*	*	*	*	*	*	*	
300	X	X	X	/	/	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	300
	X	X	X	/	/	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	
200	X	X	X	/	/	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	200
	X	/	/	*	*	/	/	/	/	/	*	*	*	*	*	*	*	*	*	/	
100	X	/	/	*	*	/	/	/	/	/	*	*	*	*	*	*	*	*	*	/	100
	X	/	/	*	*	/	/	/	/	/	*	*	*	*	/	/	/	*	*	*	
0	X	/	/	*	*	/	/	/	/	/	*	*	*	*	/	/	/	*	*	*	0

0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

Ellipticity

Figure A-4.

"Best" Method Across the Bias/Ellipticity Parameter Space

/ = RAND-234

* = Grubbs-Patnaik/chi-square

- = Grubbs-Patnaik/Wilson-Hilferty

X = Areas where the "exact" does not converge

However, there are times when the other approximation methods are more accurate than the Grubbs-Patnaik/chi-square. Which approximation technique is best varies with the bias and ellipticity of the sample impacts (bias is the square root of the sum of the squares of the means of the crossrange and downrange miss distances, ellipticity is the ratio of the crossrange standard deviation to the downrange standard deviation). Figure A-4 shows a large portion of the bias/ellipticity parameter space and which approximation techniques are "best" over the space. The technique one will ultimately use will depend on how accurately one needs to estimate the CEP of a weapon system.

It should be noted that in the regimes where the "exact" method does not converge there are other techniques that use numerical integration techniques to estimate CEP that will converge. One such method is called the correlated bivariate normal. This technique is not as efficient as the "exact" but it will converge for thos highly elliptical cases where the "exact" does not.

Conclusion

Circular error probable is not the only method of representing a weapon system's accuracy, however it is a very important measure that is widely used. Likewise the approximation methods shown here are not the only methods available for estimating CEP. This report introduces CEP and describes how it is estimated.

Attachment 1 to CEP Tutorial

Correlation

For correlated downrange and crossrange miss distances the correlation coefficient must be taken into consideration. This is the formula used to calculate the correlation coefficient:

$$\rho = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{(n - 1)(S_x S_y)} \qquad (A.1.1)$$

If the correlation coefficient is other than zero, one can perform a rotation of axes into a coordinate system where the two coordinates of miss distance are uncorrelated. This is the procedure:

$$\theta = 0.5\{\tan^{-1}[2\rho S_x S_y/(S_x{}^2 - S_y{}^2)]\} \qquad (A.1.2)$$

$$(S_x')^2 = 0.5[(S_x{}^2 + S_y{}^2) + (S_x{}^2 - S_y{}^2)/\cos(2\theta)] \qquad (A.1.3)$$

$$(S_y')^2 = 0.5[(S_x{}^2 + S_y{}^2) - (S_x{}^2 - S_y{}^2)/\cos(2\theta)] \qquad (A.1.4)$$

$$\bar{x}' = \bar{x}\cos\theta + \bar{y}\sin\theta \qquad (A.1.5)$$

$$\bar{y}' = -\bar{x}\sin\theta + \bar{y}\cos\theta \qquad (A.1.6)$$

Then S_x', S_y', $\bar{x}'$, $\bar{y}'$ are substituted for S_x, S_y, $\bar{x}$, $\bar{y}$ in the formulae and the CEP is calculated as if there were no correlation (i.e. $\rho = 0$) (3:20-21).

Astbury showed that the formulae involving a correlation coefficient are invariant to this rotation. In

other words, they give the same answer whether one uses the correlated numbers, or the uncorrelated numbers obtained from the rotation (1:3-6).

Attachment 2 to CEP Tutorial

Bibliography

1. Astbury, K. A., Ph. D, Accuracy Analysis Section. Inter-office correspondence, 85.H235.KAA-001. Subject: Computing CEP According to the Popular Approximation Formulas. TRW Defense Systems Group, Redondo Beach CA, 29 January 1985.

2. Grubbs, Frank E. "Approximate Circular and Noncircular Offset Probabilities of Hitting," Operations Research, 12 (1): 51-62 (January-February 1964).

3. Hodges, Jere and L. G. Allred. Air Force letter. Subject: Minuteman II CEP Methodology. MMEWM, Ogden AFB, Utah (no date available).

4. Pesapane, Capt John and Maj Robert B. Irvine Jr., "Derivation of CEP Formula to Approximate Rand-234 Tables." Ballistic Missile Evaluation, HQ SAC, Offutt AFB, NE, February 1977.

5. Smith, C. C. Project Engineer, Minuteman O. C. Data Base. Interoffice correspondence, 82.F651.24-008. Subject: Methods of CEP Calculation. TRW Defense and Space Systems Group, Redondo Beach CA, 26 July 1982.

6. Smith, C. C., Project Engineer, Minuteman O. C. Data Base. Interoffice correspondence, 84.H235.36-004. Subject: CEP Calculation Using Infinite Series. TRW Defense Systems Group, Redondo Beach, CA, 31 March 1984.

Appendix B

Correlation

For correlated downrange and crossrange miss distances the correlation coefficient must be taken into consideration. This is the formula used to calculate the correlation:

$$\rho = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{(n - 1)(S_x S_y)} \tag{B.1}$$

If the correlation coefficient is other than zero, one can perform a rotation of axes into a coordinate system where the two coordinates of miss distance are uncorrelated. This is the procedure:

$$\Theta = 0.5\{\tan^{-1}[2\rho S_x S_y/(S_x{}^2 - S_y{}^2)]\} \tag{B.2}$$

$$(S_{x'})^2 = 0.5[(S_x{}^2 + S_y{}^2) + (S_x{}^2 - S_y{}^2)/\cos(2\Theta)] \tag{B.3}$$

$$(S_{y'})^2 = 0.5[(S_x{}^2 + S_y{}^2) - (S_x{}^2 - S_y{}^2)/\cos(2\Theta)] \tag{B.4}$$

$$\bar{x}' = \bar{x}\cos\Theta + \bar{y}\sin\Theta \tag{B.5}$$

$$\bar{y}' = -\bar{x}\sin\Theta + \bar{y}\cos\Theta \tag{B.6}$$

Then $S_{x'}$, $S_{y'}$, $\bar{x}'$, $\bar{y}'$ are substituted for S_x, S_y, $\bar{x}$, $\bar{y}$ in the formulae and the CEP is calculated as if there is no correlation (i.e. $\rho = 0$) (3:20-21).

Astbury showed that the formulae involving a correlation coefficient are invariant to this rotation. In other words, they give the same answer whether one uses the correlated numbers, or the uncorrelated numbers obtained from the rotation. (1:3-6).

Appendix C

Secant Method

The Secant method is used to approximate the root of a function. In this case the root is the CEP (R), and the function is the probability (P(R)). The secant method takes R_i and R_{i-1} (approximations of the CEP) and takes the next approximation (R_{i+1}, $P(R_{i+1})$) as the intersection of the chord joining (R_{i-1}, $P(R_{i-1})$) and (R_i, $P(R_i)$) with the line y = 0.5 (x = R). This is the formula used:

$$R_{i+1} = R_i + (R_{i-1} - R_i)(P_{i+1} - P_i)/(P_{i-1} - P_i) \qquad (C.1)$$

where $P(R_{i+1}) = 0.5$ (I:9). Figure C-1 is a graphical representation of the secant method.

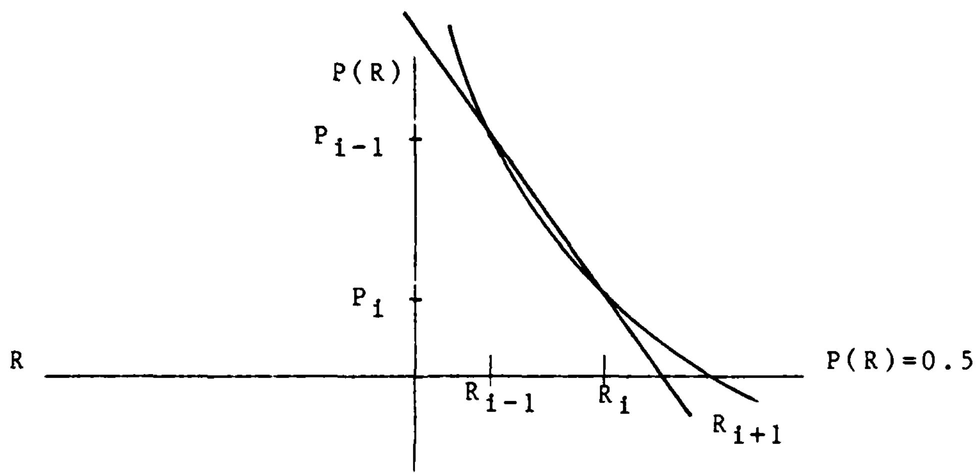

Figure C-1. The Secant Method

This method is used to iterate both the correlated bivariate normal and the "exact" approximation techniques in this study.

Appendix D

Sample Data

This appendix contains a sample of the data collected for this study. Table D-1 contains the sample cases listing the ellipticity, bias, sample standard deviations and sample means. Table D-2 is a listing of sample CEPs calculated for each of the approximation techniques for each case listed in table D-1. Finally table D-3 is the sample relative errors for the algebraic approximations and the CBN relative to the "exact" method.

CASE	ELLIP	SIG. X	SIG. Y	BIAS	X BAR	Y BAR
1	1.00	1000.00	1000.00	1555.63	1100.00	1100.00
2	1.00	1000.00	1000.00	1414.21	1000.00	1000.00
3	1.00	1000.00	1000.00	1272.79	900.00	900.00
4	1.00	1000.00	1000.00	1131.37	800.00	800.00
5	1.00	1000.00	1000.00	989.95	700.00	700.00
6	1.00	1000.00	1000.00	848.53	600.00	600.00
7	1.00	1000.00	1000.00	707.11	500.00	500.00
8	1.00	1000.00	1000.00	565.69	400.00	400.00
9	1.00	1000.00	1000.00	424.26	300.00	300.00
10	1.00	1000.00	1000.00	282.84	200.00	200.00
11	1.00	1000.00	1000.00	141.42	100.00	100.00
12	1.00	1000.00	1000.00	1.41	1.00	1.00
13	0.90	900.00	1000.00	1555.63	1100.00	1100.00
14	0.90	900.00	1000.00	1414.21	1000.00	1000.00
15	0.90	900.00	1000.00	1272.79	900.00	900.00
16	0.90	900.00	1000.00	1131.37	800.00	800.00
17	0.90	900.00	1000.00	989.95	700.00	700.00
18	0.90	900.00	1000.00	848.53	600.00	600.00
19	0.90	900.00	1000.00	707.11	500.00	500.00
20	0.90	900.00	1000.00	565.69	400.00	400.00
21	0.90	900.00	1000.00	424.26	300.00	300.00
22	0.90	900.00	1000.00	282.84	200.00	200.00
23	0.90	900.00	1000.00	141.42	100.00	100.00
24	0.90	900.00	1000.00	1.41	1.00	1.00
25	0.80	800.00	1000.00	1555.63	1100.00	1100.00
26	0.80	800.00	1000.00	1414.21	1000.00	1000.00
27	0.80	800.00	1000.00	1272.79	900.00	900.00
28	0.80	800.00	1000.00	1131.37	800.00	800.00
29	0.80	800.00	1000.00	989.95	700.00	700.00
30	0.80	800.00	1000.00	848.53	600.00	600.00
31	0.80	800.00	1000.00	707.11	500.00	500.00
32	0.80	800.00	1000.00	565.69	400.00	400.00
33	0.80	800.00	1000.00	424.26	300.00	300.00
34	0.80	800.00	1000.00	282.84	200.00	200.00
35	0.80	800.00	1000.00	141.42	100.00	100.00
36	0.80	800.00	1000.00	1.41	1.00	1.00
37	0.70	700.00	1000.00	1555.63	1100.00	1100.00
38	0.70	700.00	1000.00	1414.21	1000.00	1000.00
39	0.70	700.00	1000.00	1272.79	900.00	900.00
40	0.70	700.00	1000.00	1131.37	800.00	800.00
41	0.70	700.00	1000.00	989.95	700.00	700.00
42	0.70	700.00	1000.00	848.53	600.00	600.00
43	0.70	700.00	1000.00	707.11	500.00	500.00
44	0.70	700.00	1000.00	565.69	400.00	400.00
45	0.70	700.00	1000.00	424.26	300.00	300.00
46	0.70	700.00	1000.00	282.84	200.00	200.00
47	0.70	700.00	1000.00	141.42	100.00	100.00
48	0.70	700.00	1000.00	1.41	1.00	1.00

Table D-1
Sample Cases (part 1)

CASE	ELLIP	SIG. X	SIG. Y	BIAS	X BAR	Y BAR
49	0.60	600.00	1000.00	1555.63	1100.00	1100.00
50	0.60	600.00	1000.00	1414.21	1000.00	1000.00
51	0.60	600.00	1000.00	1272.79	900.00	900.00
52	0.60	600.00	1000.00	1131.37	800.00	800.00
53	0.60	600.00	1000.00	989.95	700.00	700.00
54	0.60	600.00	1000.00	848.53	600.00	600.00
55	0.60	600.00	1000.00	707.11	500.00	500.00
56	0.60	600.00	1000.00	565.69	400.00	400.00
57	0.60	600.00	1000.00	424.26	300.00	300.00
58	0.60	600.00	1000.00	282.84	200.00	200.00
59	0.60	600.00	1000.00	141.42	100.00	100.00
60	0.60	600.00	1000.00	1.41	1.00	1.00
61	0.50	500.00	1000.00	1555.63	1100.00	1100.00
62	0.50	500.00	1000.00	1414.21	1000.00	1000.00
63	0.50	500.00	1000.00	1272.79	900.00	900.00
64	0.50	500.00	1000.00	1131.37	800.00	800.00
65	0.50	500.00	1000.00	989.95	700.00	700.00
66	0.50	500.00	1000.00	848.53	600.00	600.00
67	0.50	500.00	1000.00	707.11	500.00	500.00
68	0.50	500.00	1000.00	565.69	400.00	400.00
69	0.50	500.00	1000.00	424.26	300.00	300.00
70	0.50	500.00	1000.00	282.84	200.00	200.00
71	0.50	500.00	1000.00	141.42	100.00	100.00
72	0.50	500.00	1000.00	1.41	1.00	1.00
73	0.40	400.00	1000.00	1555.63	1100.00	1100.00
74	0.40	400.00	1000.00	1414.21	1000.00	1000.00
75	0.40	400.00	1000.00	1272.79	900.00	900.00
76	0.40	400.00	1000.00	1131.37	800.00	800.00
77	0.40	400.00	1000.00	989.95	700.00	700.00
78	0.40	400.00	1000.00	848.53	600.00	600.00
79	0.40	400.00	1000.00	707.11	500.00	500.00
80	0.40	400.00	1000.00	565.69	400.00	400.00
81	0.40	400.00	1000.00	424.26	300.00	300.00
82	0.40	400.00	1000.00	282.84	200.00	200.00
83	0.40	400.00	1000.00	141.42	100.00	100.00
84	0.40	400.00	1000.00	1.41	1.00	1.00
85	0.30	300.00	1000.00	1555.63	1100.00	1100.00
86	0.30	300.00	1000.00	1414.21	1000.00	1000.00
87	0.30	300.00	1000.00	1272.79	900.00	900.00
88	0.30	300.00	1000.00	1131.37	800.00	800.00
89	0.30	300.00	1000.00	989.95	700.00	700.00
90	0.30	300.00	1000.00	848.53	600.00	600.00
91	0.30	300.00	1000.00	707.11	500.00	500.00
92	0.30	300.00	1000.00	565.69	400.00	400.00
93	0.30	300.00	1000.00	424.26	300.00	300.00
94	0.30	300.00	1000.00	282.84	200.00	200.00
95	0.30	300.00	1000.00	141.42	100.00	100.00
96	0.30	300.00	1000.00	1.41	1.00	1.00

Table D-I
Sample Cases (part 2)

CASE	ELLIP	SIG. X	SIG. Y	BIAS	X BAR	Y BAR
97	0.20	200.00	1000.00	1555.63	1100.00	1100.00
98	0.20	200.00	1000.00	1414.21	1000.00	1000.00
99	0.20	200.00	1000.00	1272.79	900.00	900.00
100	0.20	200.00	1000.00	1131.37	800.00	800.00
101	0.20	200.00	1000.00	989.95	700.00	700.00
102	0.20	200.00	1000.00	848.53	600.00	600.00
103	0.20	200.00	1000.00	707.11	500.00	500.00
104	0.20	200.00	1000.00	565.69	400.00	400.00
105	0.20	200.00	1000.00	424.26	300.00	300.00
106	0.20	200.00	1000.00	282.84	200.00	200.00
107	0.20	200.00	1000.00	141.42	100.00	100.00
108	0.20	200.00	1000.00	1.41	1.00	1.00
109	0.10	100.00	1000.00	1555.63	1100.00	1100.00
110	0.10	100.00	1000.00	1414.21	1000.00	1000.00
111	0.10	100.00	1000.00	1272.79	900.00	900.00
112	0.10	100.00	1000.00	1131.37	800.00	800.00
113	0.10	100.00	1000.00	989.95	700.00	700.00
114	0.10	100.00	1000.00	848.53	600.00	600.00
115	0.10	100.00	1000.00	707.11	500.00	500.00
116	0.10	100.00	1000.00	565.69	400.00	400.00
117	0.10	100.00	1000.00	424.26	300.00	300.00
118	0.10	100.00	1000.00	282.84	200.00	200.00
119	0.10	100.00	1000.00	141.42	100.00	100.00
120	0.10	100.00	1000.00	1.41	1.00	1.00
121	0.01	10.00	1000.00	1555.63	1100.00	1100.00
122	0.01	10.00	1000.00	1414.21	1000.00	1000.00
123	0.01	10.00	1000.00	1272.79	900.00	900.00
124	0.01	10.00	1000.00	1131.37	800.00	800.00
125	0.01	10.00	1000.00	989.95	700.00	700.00
126	0.01	10.00	1000.00	848.53	600.00	600.00
127	0.01	10.00	1000.00	707.11	500.00	500.00
128	0.01	10.00	1000.00	565.69	400.00	400.00
129	0.01	10.00	1000.00	424.26	300.00	300.00
130	0.01	10.00	1000.00	282.84	200.00	200.00
131	0.01	10.00	1000.00	141.42	100.00	100.00
132	0.01	10.00	1000.00	1.41	1.00	1.00

Table D-1
Sample Cases (part 3)

CASE	Grubbs/CS	Grubbs/WH	RAND	CBN	Exact
1	1857.29	1861.86	1867.99	1850.30	1869.75
2	1750.71	1755.28	1758.27	1740.69	1652.25
3	1650.01	1654.75	1655.09	1637.23	1654.24
4	1556.27	1561.32	1559.44	1541.35	1557.31
5	1470.70	1476.14	1472.28	1454.52	1469.56
6	1394.53	1400.35	1394.59	1378.04	1392.28
7	1328.91	1335.03	1327.33	1312.84	1326.38
8	1274.75	1281.04	1271.48	1259.46	1272.47
9	1232.61	1238.95	1228.01	1218.10	1230.69
10	1202.70	1208.99	1197.89	1188.72	1201.00
11	1184.89	1191.12	1182.09	1171.18	1183.28
12	1178.98	1185.19	1181.52	1165.35	1177.38
13	1825.38	1829.81	1836.14	1820.39	1839.29
14	1715.94	1720.28	1723.07	1707.95	1669.30
15	1612.11	1616.56	1616.36	1601.20	1555.53
16	1515.04	1519.76	1517.07	1501.66	1517.38
17	1426.00	1431.10	1426.29	1410.96	1425.86
18	1346.36	1351.88	1345.11	1330.64	1344.82
19	1277.44	1283.31	1274.59	1261.90	1275.46
20	1220.34	1226.45	1215.83	1205.50	1218.55
21	1175.84	1182.01	1169.91	1161.75	1174.39
22	1144.46	1150.34	1137.90	1130.67	1143.00
23	1125.69	1131.43	1120.88	1112.11	1124.27
24	1119.45	1125.16	1119.88	1105.94	1118.04
25	1795.71	1799.82	1806.21	1791.28	1809.55
26	1683.10	1687.27	1689.80	1676.48	1643.36
27	1576.01	1580.23	1579.45	1566.84	1583.02
28	1475.35	1479.84	1476.35	1463.87	1416.66
29	1382.49	1387.41	1381.73	1369.27	1383.93
30	1298.90	1304.33	1296.79	1284.81	1298.86
31	1226.11	1232.01	1222.74	1212.02	1225.53
32	1165.71	1171.72	1160.80	1151.97	1165.03
33	1118.96	1124.38	1112.18	1105.25	1117.93
34	1085.45	1090.52	1078.08	1071.99	1084.38
35	1065.37	1070.28	1059.72	1052.12	1064.34
36	1058.70	1063.55	1058.24	1045.51	1057.67
37	1768.41	1772.17	1778.26	1762.20	1779.77
38	1652.56	1656.61	1658.58	1645.52	1618.01
39	1542.11	1546.17	1544.53	1533.56	1549.18
40	1437.69	1442.04	1437.48	1427.66	1442.56
41	1340.69	1345.55	1338.78	1329.46	1280.21
42	1252.70	1258.23	1249.81	1240.81	1254.65
43	1175.51	1181.65	1171.93	1163.56	1176.99
44	1112.22	1117.32	1106.50	1099.23	1112.28
45	1062.04	1066.47	1054.90	1048.80	1061.52
46	1025.85	1029.88	1018.48	1012.74	1025.20
47	1004.08	1007.90	998.62	991.14	1003.44
48	996.82	1000.59	996.60	983.95	996.19

Table D-2
Sample CEPs (part 1)

CASE	Grubbs/CS	Grubbs/WH	RAND	CBN	Exact
49	1743.70	1747.20	1752.28	1732.18	1749.07
50	1624.73	1628.68	1629.51	1614.08	1592.31
51	1510.93	1514.88	1511.78	1500.40	1515.39
52	1402.66	1406.94	1400.65	1392.25	1406.60
53	1301.30	1306.23	1297.65	1291.09	1304.93
54	1208.51	1214.36	1204.36	1198.64	1146.17
55	1127.88	1133.08	1122.30	1116.87	1130.13
56	1060.05	1064.15	1053.05	1047.72	1060.74
57	1005.74	1009.14	998.14	992.79	1005.58
58	966.23	969.22	959.14	953.13	965.71
59	942.30	945.09	937.58	929.22	941.67
60	934.30	937.03	934.96	921.25	933.64
61	1721.93	1725.24	1728.11	1700.22	1716.59
62	1600.06	1603.94	1602.62	1531.01	1565.18
63	1483.04	1486.91	1481.35	1466.07	1480.50
64	1370.97	1375.24	1366.08	1356.35	1370.16
65	1265.19	1270.30	1258.59	1253.05	1266.45
66	1167.81	1173.76	1160.67	1157.63	1170.78
67	1083.23	1087.50	1074.07	1071.87	1013.89
68	1010.35	1013.54	1000.60	997.88	1010.84
69	951.25	953.85	942.01	937.85	950.74
70	907.75	910.08	900.10	893.70	906.50
71	881.15	883.40	876.63	866.75	879.48
72	872.21	874.44	873.32	857.70	870.40
73	1703.48	1706.65	1705.39	1665.27	1706.65
74	1579.03	1582.85	1577.81	1545.09	1535.14
75	1459.04	1462.86	1453.33	1429.19	1443.34
76	1343.41	1347.72	1333.99	1318.40	1331.93
77	1233.38	1238.78	1221.87	1213.71	1226.77
78	1132.56	1137.69	1119.01	1116.30	1129.14
79	1042.99	1046.46	1027.48	1027.65	1040.45
80	964.77	967.35	949.33	949.67	881.86
81	900.40	902.73	886.63	884.74	897.83
82	852.33	854.79	841.43	835.62	848.82
83	822.60	825.29	815.78	804.92	818.21
84	812.54	815.34	811.68	794.47	807.79
85	1688.69	1691.76	1683.33	1626.27	1665.70
86	1562.08	1565.87	1554.75	1505.00	1544.35
87	1439.57	1443.36	1427.69	1388.25	1437.18
88	1320.83	1325.22	1304.56	1388.25	1307.55
89	1207.01	1212.01	1187.76	1172.30	1205.28
90	1103.13	1107.56	1079.71	1075.52	1085.98
91	1008.86	1011.76	982.82	973.40	994.33
92	925.43	927.81	899.50	888.73	915.50
93	855.73	858.44	832.16	821.45	745.28
94	802.87	806.41	783.21	774.59	794.54
95	769.73	774.10	755.06	740.13	760.92
96	758.43	763.14	750.04	738.92	749.96

Table D-2
Sample CEPs (part 2)

CASE	Grubbs/CS	Grubbs/WH	RAND	CBN	Exact
97	1677.88	1680.89	1660.44	1585.93	*
98	1549.65	1553.40	1532.65	1462.39	*
99	1425.18	1428.96	1404.17	1344.02	*
100	1304.02	1308.49	1277.84	1231.65	*
101	1187.17	1193.22	1156.54	1125.89	*
102	1080.84	1084.78	1043.13	1027.02	*
103	982.67	985.27	940.46	936.56	*
104	894.78	897.30	851.40	856.12	870.61
105	820.35	823.93	778.81	788.00	802.83
106	763.08	768.39	725.56	735.30	607.95
107	726.72	733.65	694.49	701.57	717.34
108	714.23	721.81	688.40	689.91	705.77
109	1671.29	1674.26	1633.90	1560.22	1658.24
110	1542.04	1545.78	1509.86	1433.68	*
111	1416.35	1420.12	1381.96	1312.60	*
112	1293.63	1298.17	1253.66	1198.19	*
113	1175.02	1181.12	1128.40	1091.35	*
114	1066.91	1070.56	1009.65	992.36	*
115	966.13	968.61	900.84	902.83	*
116	875.20	877.98	805.43	824.15	*
117	797.44	801.93	726.87	758.36	*
118	736.99	744.01	668.62	708.13	*
119	698.26	707.60	634.12	676.38	411.30
120	684.88	695.16	626.76	665.49	615.00
121	1669.09	1672.06	1602.61	1667.58	*
122	1539.51	1543.24	1486.09	1572.47	*
123	1413.39	1417.17	1361.66	1402.40	*
124	1290.15	1294.72	1233.47	1319.39	*
125	1171.07	1177.05	1105.68	1192.82	*
126	1062.22	1065.77	982.45	1066.58	*
127	960.53	962.98	867.93	977.34	*
128	868.51	871.43	766.27	863.73	*
129	789.57	794.43	681.63	751.08	*
130	727.97	735.68	618.16	817.96	*
131	688.36	698.68	580.02	774.81	*
132	674.64	686.03	571.29	570.98	*

Table D-2
Sample CEPs (part 3)

Note: * = "exact" does not converge

CASE	Grubbs/CS	Grubbs/WH	RAND	CBN	"BEST"
1	-0.0067	-0.0042	-0.0009	-0.0104	3
2	0.0596	0.0624	0.0642	0.0535	1
3	-0.0026	0.0003	0.0005	-0.0103	2
4	-0.0007	0.0026	0.0014	-0.0102	1
5	0.0008	0.0045	0.0019	-0.0102	1
6	0.0016	0.0058	0.0017	-0.0102	1
7	0.0019	0.0065	0.0007	-0.0102	3
8	0.0018	0.0067	-0.0008	-0.0102	3
9	0.0016	0.0067	-0.0022	-0.0102	1
10	0.0014	0.0067	-0.0026	-0.0102	1
11	0.0014	0.0066	-0.0010	-0.0102	3
12	0.0014	0.0066	0.0035	-0.0102	1
13	-0.0076	-0.0052	-0.0017	-0.0103	3
14	0.0279	0.0305	0.0322	0.0232	1
15	0.0364	0.0392	0.0391	0.0294	1
16	-0.0015	0.0016	-0.0002	-0.0104	3
17	0.0001	0.0037	0.0003	-0.0104	1
18	0.0011	0.0053	0.0002	-0.0105	3
19	0.0015	0.0062	-0.0007	-0.0106	3
20	0.0015	0.0065	-0.0022	-0.0107	1
21	0.0012	0.0065	-0.0038	-0.0108	1
22	0.0013	0.0064	-0.0045	-0.0108	1
23	0.0013	0.0064	-0.0030	-0.0108	1
24	0.0013	0.0064	0.0016	-0.0108	1
25	-0.0076	-0.0054	-0.0018	-0.0101	3
26	0.0242	0.0267	0.0283	0.0202	1
27	-0.0044	-0.0018	-0.0023	-0.0102	2
28	0.0414	0.0446	0.0421	0.0333	1
29	-0.0010	0.0025	-0.0016	-0.0106	1
30	0.0000	0.0042	-0.0016	-0.0108	1
31	0.0005	0.0053	-0.0023	-0.0110	1
32	0.0006	0.0057	-0.0036	-0.0112	1
33	0.0009	0.0058	-0.0051	-0.0113	1
34	0.0010	0.0057	-0.0058	-0.0114	1
35	0.0010	0.0056	-0.0043	-0.0115	1
36	0.0010	0.0056	0.0005	-0.0115	3
37	-0.0064	-0.0043	-0.0008	-0.0099	3
38	0.0214	0.0239	0.0251	0.0170	1
39	-0.0046	-0.0019	-0.0030	-0.0101	2
40	-0.0034	-0.0004	-0.0035	-0.0103	2
41	0.0472	0.0510	0.0457	0.0385	3
42	-0.0016	0.0028	-0.0039	-0.0110	1
43	-0.0013	0.0040	-0.0043	-0.0114	1
44	-0.0000	0.0045	-0.0052	-0.0117	1
45	0.0005	0.0047	-0.0062	-0.0120	1
46	0.0006	0.0046	-0.0066	-0.0122	1
47	0.0006	0.0044	-0.0048	-0.0123	1
48	0.0006	0.0044	0.0004	-0.0123	3

Table D-3
Sample Relative Errors (part 1)

CASE	Grubbs/CS	Grubbs/WH	RAND	CBN	"BEST"
49	-0.0031	-0.0011	0.0018	-0.0097	2
50	0.0204	0.0229	0.0234	0.0137	1
51	-0.0029	-0.0003	-0.0024	-0.0099	2
52	-0.0028	0.0002	-0.0043	-0.0102	2
53	-0.0028	0.0010	-0.0056	-0.0106	2
54	0.0544	0.0595	0.0508	0.0458	3
55	-0.0020	0.0026	-0.0069	-0.0117	1
56	-0.0006	0.0032	-0.0072	-0.0123	1
57	0.0002	0.0036	-0.0073	-0.0126	1
58	0.0006	0.0037	-0.0068	-0.0130	1
59	0.0007	0.0036	-0.0043	-0.0132	1
60	0.0007	0.0036	0.0014	-0.0133	1
61	0.0031	0.0050	0.0067	-0.0095	1
62	0.0223	0.0248	0.0239	0.0101	1
63	0.0017	0.0043	0.0006	-0.0097	3
64	0.0006	0.0037	-0.0030	-0.0101	1
65	-0.0010	0.0030	-0.0062	-0.0106	1
66	-0.0025	0.0025	-0.0086	-0.0112	1
67	0.0684	0.0726	0.0594	0.0572	3
68	-0.0005	0.0027	-0.0101	-0.0128	1
69	0.0005	0.0033	-0.0092	-0.0136	1
70	0.0014	0.0040	-0.0071	-0.0141	1
71	0.0019	0.0045	-0.0032	-0.0145	1
72	0.0021	0.0046	0.0034	-0.0146	1
73	-0.0019	0.0000	-0.0007	-0.0242	2
74	0.0286	0.0311	0.0278	0.0065	3
75	0.0109	0.0135	0.0069	-0.0098	3
76	0.0086	0.0119	0.0016	-0.0102	3
77	0.0054	0.0098	-0.0040	-0.0106	3
78	0.0030	0.0076	-0.0090	-0.0114	1
79	0.0024	0.0058	-0.0125	-0.0123	1
80	0.0940	0.0969	0.0765	0.0769	3
81	0.0029	0.0055	-0.0125	-0.0146	1
82	0.0041	0.0070	-0.0087	-0.0156	1
83	0.0054	0.0087	-0.0030	-0.0162	3
84	0.0059	0.0093	0.0048	-0.0165	3
85	0.0138	0.0156	0.0106	-0.0237	3
86	0.0115	0.0139	0.0067	-0.0255	3
87	0.0017	0.0043	-0.0066	-0.0340	1
88	0.0102	0.0135	-0.0023	-1.0000	3
89	0.0014	0.0056	-0.0145	-0.0027	1
90	0.0158	0.0199	-0.0058	-0.0096	3
91	0.0146	0.0175	-0.0116	-0.0210	3
92	0.0108	0.0134	-0.0175	-0.0292	1
93	0.1482	0.1518	0.1166	0.1022	3
94	0.0105	0.0149	-0.0143	-0.0251	1
95	0.0116	0.0173	-0.0077	-0.0273	3
96	0.0107	0.0176	0.0001	-0.0147	3

Table D-3
Sample Relative Errors (part 2)

CASE	Grubbs/CS	Grubbs/WH	RAND	CBN	"BEST"
97					
|					
103					
104	0.0278	0.0307	-0.0221	-0.0166	3
105	0.0218	0.0263	-0.0058	-0.0096	3
106	0.2552	0.2639	0.1934	0.2095	3
107	0.0131	0.0227	-0.0319	-0.0220	1
108	0.0120	0.0227	-0.0246	-0.0225	1
109	0.0079	0.0097	-0.0147	-0.0591	1
110					
|					
118					
119	0.6983	0.7202	0.5426	0.6448	3
120	0.1136	0.1303	0.0191	0.0821	3
121					
|					
132					

Table D-3
Sample Relative Errors (part 3)

Note: no relative errors where "exact" does not converge

Appendix E

This appendix contains the source code for the computer program developed by this study. The program is written in Turbo Pascal (Turbo Pascal is a product of Borland International), a version of the Pascal computer programming language for personal computers. The program was written and run on a SANYO MBC-550 personal computer (operating system: MS-DOS) owned by the author. This program was also run on a Zenith Z-150 personal computer at the Air Force Institute of Technology. With some minor alterations this program can be run on any computer with any version of the Pascal programming language.

This program runs interactively with the user inputs to calculate circular error probable (CEP) using five different approximation techniques. The program will calculate CEP for all five methods, only the methods the user selects, or it will select a "best" method from among the three algebraic methods which give results in seconds. The user decides which method(s) the program will compute.

{THIS PROGRAM WAS DEVELOPED BY CAPT RICHARD L. ELDER AS A PART OF A MASTER'S THESIS AT THE AIR FORCE INSTITUTE OF TECHNOLOGY. TITLE OF THE THESIS IS "AN EXAMINATION OF CIRCULAR ERROR PROBABLE APPROXIMATION TECHNIQUES."}

```
program     CEPAPPROXIMATIONANALYSIS ;

    var sigx, sigy, mux, muy, rho, ellip, bias:  real;
        cbnprob, randc, gpcs, gpwh, prob, cep, cbn:  real;
        pick, ints:  integer;

                                    **********

  function atotheb(a,b:  real):  real;

     {This function takes an input variable a and raises it
          to the b power}

    begin
      if a = 0 then atotheb:= 0
        else if (b*ln(a)) < -70 then atotheb:= 0
               else atotheb:= exp(b*ln(a))
    end  {atotheb};

                                    **********

  function interpolate(a, aup, alow, kup, klow:  real):  real;

     {This function is a function to perform linear
          interpolation.  It is used in procedure GPCSCEP
          to interpolate between values found in a table
          of the chi-square function.}

    var percent:  real;

    begin
      percent:= (a - alow)/(aup - alow);
      interpolate:= klow + (percent*(kup - klow))
    end  {interpolate};

                                    **********
```

```
procedure GPCSCEP(var sigx, sigy, mux, muy, rho:  real);

   {This procedure calculates CEPs according to the
        Grubbs-Patnaik/chi-square approximation technique}

  var m, x, y, v, n, k:  real;

  begin
    m:= sqr(sigx)+sqr(sigy)+sqr(mux)+sqr(muy);
    x:= atotheb(sigx, 4);
    y:= atotheb(sigy, 4);
    v:= 2*(x+(2*sqr(rho*sigx*sigy))+y);
    v:= v+(4*(sqr(mux*sigx)+(2*mux*muy*rho*sigx*sigy)+sqr(muy*sigy)));
    n:= 2*sqr(m)/v;
    if n >= 11 then k:= (n-0.7);
    case trunc(n) of
      0: k:= interpolate(n,1,0,0.455,0);
      1: k:= interpolate(n,2,1,1.39,0.455);
      2: k:= interpolate(n,3,2,2.37,1.39);
      3: k:= interpolate(n,4,3,3.36,2.37);
      4: k:= interpolate(n,5,4,4.35,3.36);
      5: k:= interpolate(n,6,5,5.35,4.35);
      6: k:= interpolate(n,7,6,6.35,5.35);
      7: k:= interpolate(n,8,7,7.34,6.35);
      8: k:= interpolate(n,9,8,8.34,7.34);
      9: k:= interpolate(n,10,9,9.34,8.34);
      10: k:= interpolate(n,11,10,10.3,9.34)
    end;
    gpcs:= sqrt((k*v)/(2*m))
  end  {GPCSCEP};
```

****

```
procedure GPWHCEP(var sigx, sigy, mux, muy:  real);

   {This procedure calculates CEPs according to the
        Grubbs-Patnaik/Wilson-Hilferty approximation
        technique.}

  var aa, m, v, vv, vvv:  real;

  begin
    m:= sqr(sigx) + sqr(sigy) + sqr(mux) + sqr(muy);
    vv:= sqr(sqr(sigx)) + 2* rho * sqr(sigx) * sqr(sigy) + sqr(sqr(sigy));
    vvv:= sqr(mux)*sqr(sigx) + 2*mux*muy*rho*sigx*sigy + sqr(muy)*sqr(sigy);
    v:= 2*vv + 4*vvv;
    aa:= 1 - (v/(9*sqr(m)));
    gpwh:= sqrt(m*(atotheb(aa, 3)));
  end {GPWHCEP};
```

****

```
procedure RANDCEP (var sigx, sigy, mux, muy:  real);

   {This procedure calculates CEPs according to the
        modified RAND-234 approximation technique}

  var radical, sigxy, sigS, sigL, b, v, cepmu:  real;

  begin
    sigxy:= sqr(sigx) + sqr(sigy);
    radical:= sqrt(sqr(sqr(sigx) - sqr(sigy)) + 4*sqr(rho*sigx*sigy));
    sigS:= sqrt((sigxy - radical)/2);
    sigL:= sqrt((sigxy + radical)/2);
    cepmu:= 0.614*sigS + 0.563*sigL;
    b:= sqrt(sqr(mux) + sqr(muy));
    v:= b/cepmu;
    randc:= cepmu*(1.0039-0.0528*v+0.4786*sqr(v)-0.0793*atotheb(v,3))
  end; {RANDCEP}
```

****

```
   {The following procedure is called by the CBNORM
        procedure which calculates the CEPs for the
        correlated bivariate normal (CBN) approximation
        technique.  This procedure calculates the probability
        given a known radius (an estimate of CEP).  It is
        called until the probability is within 0.0001 of
        0.5.  The accuracy of this method can be improved
        by increasing the number of intervals used to
        approximate the integral.  This number of intervals
        is a user input that is asked for whenever the CBN
        method is used.
```

```
procedure  CBNORMPROB(var sigx, sigy, mux, muy, rho, bias:  real;
                          ints:  integer);

  label 1;

  var y, hh, i, x, c, t, a: real;
      jj, j, h:  integer;

  begin
    c:= 2*3.1415927*sigy*sigx*sqrt(1-sqr(rho));
    c:= 1/c;
    i:= bias;
    t:= 0;
    for j := 0 to ints
      do begin
        x:= -i + (j*(i/(ints/2)));
        y:= sqrt(abs(sqr(i) - sqr(x)));
        if y = 0 then goto 1;
        for h := 0 to ints
          do begin
            hh:= -y + (h*(y/(ints/2)));
            a:= sqr((x-mux)/sigx)-(2*rho*((x-mux)/sigx)*((hh-muy)/sigy)) +
                sqr((hh-muy)/sigy);
            a:= a/(1-sqr(rho));
            if a > 140 then a:= 0
              else a:= exp(-0.5*a);
            a:= a*c;
            t:= t + (a*(i/(ints/2))*2*(y/ints))
          end;
1:      jj:= j
      end;
      cbnprob:= t;
  end;
```

****

```
{The following procedure is the secant method applied
     to the correlated bivariate normal approximation
     technique.  The stopping criterion for this method
     is when the probability given by the CBNORMPROB
     procedure is within 0.0001 of 0.5}
```

```
procedure CBNORM (var sigx, sigy, mux, muy, rho, cep:  real;
                      ints);

  var cepipl, cepi, cepiml, probipl, probi, probiml: real;

  begin
    cepiml:= cep;
    CBNORMPROB(sigx, sigy, mux, muy, rho, cepiml, ints);
    probiml:= cbnprob;
    if cbnprob < 0.5 then cepi:= 1.05*cep
      else cepi:= 0.95*cep;
    CBNORMPROB(sigx, sigy, mux, muy, rho, cepi, ints);
    probi:= cbnprob;
    probipl:= 0.5;
    while abs(cbnprob - 0.5) > 0.0001 do
      begin
        cepipl:= cepi+((cepiml-cepi)*(probipl-probi)/(probiml-probi));
        CBNORMPROB(sigx, sigy, mux, muy, rho, cepipl, ints);
        cepiml:= cepi;
        probiml:= probi;
        cepi:= cepipl;
        probi:= cbnprob;
      end;
    cbn:= cepipl
  end;
```

****

```
   {The following procedure is called by EXACT which
        calculates CEPs for the "exact" method.  EXACTPROB
        calculates probabilities given an estimate of the
        CEP.  It is called until the probability is within
        0.0001 of 0.5.}

procedure EXACTPROB (var sigx, sigy, mux, muy, cep:  real);

type  logarray = array[1..100] of real;

  var kcheck, jcheck, Lsum, kf, kmLf, L2f, hold, La, Lterm:  real;
      k2f, kplf, xka, xk, isum, jf, jmif, i2f, ia, iterm:  real;
      j2f, jplf, yja, yj, kpjplf, jterm, jsum, ksum, d:  real;
      r, ja, yjb, xkb, Lb, ib, Lc, ic:  real;
      log:  logarray;
      i, j, k, L, il, i2, jl, j2, j3, j4, kl, k2, k3, Ll, L2:  integer;
```

```
begin {EXACTPROB}
  r:= cep;
  kcheck:= 1;
  k:= 0;
  ksum:= 0;
  while kcheck > 0.0001 do
    begin
      kf:= 0;
      if k > 0 then for k1:= 1 to k do kf:= kf + log[k1];
      k2f:= 0;
      if k > 0 then for k2:= 1 to (2*k) do k2f:= k2f + log[k2];
      kp1f:= 0;
      for k3:= 1 to (k+1) do kp1f:= kp1f + log[k3];
      jcheck:= 1;
      j:= 0;
      jsum:= 0;
      while jcheck > 0.0001 do
        begin
          writeln(k, '  ', kcheck, j, '  ', jcheck);
          jf:= 0;
          if j > 0 then for j1:= 1 to j do jf:= jf + log[j1];
          jp1f:= 0;
          for j2:= 1 to (j+1) do jp1f:= jp1f + log[j2];
          j2f:= 0;
          if j > 0 then for j3:= 1 to (2*j) do j2f:= j2f + log[j3];
          kpjp1f:= 0;
          for j4:= 1 to (k+j+1) do kpjp1f:= kpjp1f + log[j4];
          Lsum:= 0;
          for L:= 0 to k do
            begin
              kmLf:= 0;
              if (k-L) > 0 then for L1:= 1 to (k-L) do kmLf:= kmLf+log[L1];
              L2f:= 0;
              if L > 0 then for L2:= 1 to (2*L) do L2f:= L2f + log[L2];
              hold:= (2*sqr(mux))/(sqr(sigx));
              La:= atotheb(hold, (L/2));
              Lc:= La;
              if odd(L) then La:= -La;
              if (kf-kmLf-L2f) < -70 then Lb:= 0
                else Lb:= exp(kf - kmLf - L2f);
              if (abs(La) < 1.0e-16) or (abs(Lb) < 1.0e-16) then Lterm:= 0
                else Lterm:= La*Lb*Lc;
              Lsum:= Lsum + Lterm
            end;
```

{EXACTPROB continued}

```
              hold:= sqr(r)/(8*(sqr(sigx)));
              xka:= atotheb(hold, k);
              if odd(k) then xka:= -xka;
              if (k2f-kplf-(2*kf)) < -70 then xkb:= 0
                else xkb:= exp(k2f - kplf - (2*kf));
              xk:= xka*xkb*Lsum;
              isum:= 0;
              for i:= 0 to j do
                begin
                  jmif:= 0;
                  if (j-i) > 0 then for i1:= 1 to (j-i) do jmif:= jmif+log[i1];
                  i2f:= 0;
                  if i > 0 then for i2:= 1 to (2*i) do i2f:= i2f + log[i2];
                  hold:= (2*(sqr(muy)))/(sqr(sigy));
                  ia:= atotheb(hold, (i/2));
                  ic:= ia;
                  if odd(i) then ia:= -ia;
                  if (jf-jmif-i2f) < -70 then ib:= 0
                    else  ib:= exp(jf - jmif - i2f);
                  if (abs(ia) < 1.0e-16) or (abs(ib) < 1.0e-16) then iterm:= 0
                    else iterm:= ia*ib*ic;
                  isum:= isum + iterm
                end;
              hold:= (sqr(r))/(8*(sqr(sigy)));
              yja:= atotheb(hold, j);
              if odd(j) then yja:= -yja;
              if (j2f-jplf-(2*jf)) < -70 then yjb:= 0
                else yjb:= exp(j2f- jplf - (2*jf));
              yj:= yja*yjb*isum;
              if (kplf+jplf-kpjplf) < -70 then ja:= 0
                else ja:= exp(kplf + jplf - kpjplf);
              jterm:= xk*yj*ja;
              jsum:= jsum + jterm;
              jcheck:= abs(jterm);
              j:= j + 1
            end;   {while jcheck ... }
          ksum:= ksum + jsum;
          kcheck:= abs(jsum);
          k:= k + 1
        end  {while kcheck ... };
      d:= (exp(-(sqr(mux)/(2*sqr(sigx)))-(sqr(muy)/(2*sqr(sigy)))))/(sigx*sigy);
      prob:= (sqr(r)/2)*d*ksum
    end;   {EXACTPROB}
```

****

{The following procedure is the secant method applied to the "exact" approximation technique. It calls the procedure EXACTPROB until the probability is within 0.0001 of 0.5.}

```
procedure EXACT (var sigx, sigy, mux, muv, cep:  real);

  var cepipl, cepi, cepiml, probipl, probi, probiml: real;

  begin
    for ii:= 1 to 100 do log[ii]:= ln(ii);
    cepiml:= cep;
    EXACTPROB(sigx, sigy, mux, muy, cepiml);
    probiml:= prob;
    if prob < 0.5 then cepi:= 1.05*cep
      else cepi:= 0.95*cep;
    EXACTPROB(sigx, sigy, mux, muy, cepi);
    probi:= prob;
    probipl:= 0.5;
    while abs(prob - 0.5) > 0.0001 do
      begin
        cepipl:= cepi+((cepiml-cepi)*(probipl-probi)/(probiml-probi));
        EXACTPROB(sigx, sigy, mux, muy, cepipl);
        cepiml:= cepi;
        probiml:= probi;
        cepi:= cepipl;
        probi:= prob;
      end;
    cep:= cepipl
  end;
```

****

```
 {The following procedure is the decision maker for which
      of the algebraic approximations is the "best" for the
      given regime of the ellipticity/bias parameter space.}
```

```
procedure BESTALGEBRAIC(var ints:  integer;
                             sigx, sigy, mux, muy, rho, bias, ellip:  real);

  begin
    if ellip < 0.05 then writeln('SORRY, YOU MUST CHOOSE A METHOD.');
    if (ellip < 0.15) and (ellip >= 0.05)
      then begin
        if bias <= 200
          then begin
            RANDCEP(sigx, sigy, mux, muy);
            write('THE MODIFIED RAND-234 IS BEST HERE:  ');
            writeln(randc:10:3)
          end;
        if bias > 200 then writeln('SORRY, YOU MUST CHOOSE A METHOD.')
      end;
    if (ellip < 0.25) and (ellip >= 0.15)
      then begin
        if (bias < 200) or ((bias >= 350) and (bias < 500))
          then begin
            GPCSCEP(sigx, sigy, mux, muy, rho);
            write('THE GRUBBS-PATNAIK/CHI-SQUARE IS BEST HERE:  ');
            writeln(gpcs:10:3)
          end;
        if ((bias >= 200) and (bias < 350)) or
           ((bias >= 500) and (bias <= 650))
          then begin
            RANDCEP(sigx, sigy, mux, muy);
            write('THE MODIFIED RAND-234 IS BEST HERE:  ');
            writeln(randc:10:3)
          end;
        if bias > 650 then writeln('SORRY, YOU MUST CHOOSE A METHOD.')
      end;
    if (ellip < 0.5) and (ellip >= 0.25) and (bias < 200)
      then begin
        RANDCEP(sigx, sigy, mux, muy);
        writeln('THE MODIFIED RAND-234 IS BEST HERE:  ', randc:10:3)
      end;
```

{BESTALGEBRAIC continued}

```
 if (ellip < 0.4) and (ellip >= 0.25)
   then begin
     if ((bias >= 200) and (bias < 350)) or
        ((bias >= 500) and (bias < 650)) or
        ((bias >= 900) and (bias < 1000)) or
        ((bias >= 1200) and (bias < 1300))
        then begin
          GPCSCEP(sigx, sigy, mux, muy, rho);
          write('THE GRUBBS-PATNAIK/CHI-SQUARE IS BEST HERE:  ');
          writeln(gpcs:10:3)
        end;
     if ((bias >= 350) and (bias < 500)) or
        ((bias >= 650) and (bias < 900)) or
        ((bias >= 1000) and (bias < 1200)) or
        ((bias >= 1300) and (bias <= 1550))
        then begin
          RANDCEP(sigx, sigy, mux, muy);
          write('THE MODIFIED RAND-234 IS BEST HERE:  ');
          writeln(randc:10:3)
        end
   end;
 if (ellip >= 0.4) and (ellip < 0.5)
   then begin
     if ((bias >= 200) and (bias < 500)) or
        ((bias >= 600) and (bias < 900)) or
        ((bias >= 1450) and (bias <= 1550))
        then begin
          GPCSCEP(sigx, sigy, mux, muy, rho);
          write('THE GRUBBS-PATNAIK/CHI-SQUARE IS BEST HERE:  ');
          writeln(gpcs:10:3)
        end;
     if ((bias >= 500) and (bias < 600)) or
        ((bias >= 900) and (bias < 1450))
        then begin
          RANDCEP(sigx, sigy, mux, muy);
          write('THE MODIFIED RAND-234 IS BEST HERE:  ');
          writeln(randc:10:3)
        end
   end;
 if (ellip >= 0.5) and (ellip < 0.7) and
    ((bias < 600) or ((bias >= 1300) and (bias < 1450)))
    then begin
      GPCSCEP(sigx, sigy, mux, muy, rho);
      write('THE GRUBBS-PATNAIK/CHI-SQUARE IS BEST HERE:  ');
      writeln(gpcs:10:3)
    end;
```

```
{BESTALGEBRAIC continued}

 if (ellip >= 0.5) and (ellip < 0.6)
   then begin
     if ((bias >= 600) and (bias < 750)) or
        ((bias >= 1200) and (bias < 1300))
        then begin
          RANDCEP(sigx, sigy, mux, muy);
          write('THE MODIFIED RAND-234 IS BEST HERE:  ');
          writeln(randc:10:3)
        end;
     if ((bias >= 750) and (bias < 1200)) or
        ((bias >= 1450) and (bias < 1550))
        then begin
          GPCSCEP(sigx, sigy, mux, muy, rho);
          write('THE GRUBBS-PATNAIK/CHI-SQUARE IS BEST HERE:  ');
          writeln(gpcs:10:3)
        end
   end;
 if (ellip >= 0.6) and (ellip < 0.7)
   then begin
     if (bias >= 750) and (bias < 900)
       then begin
         RANDCEP(sigx, sigy, mux, muy);
         write('THE MODIFIED RAND-234 IS BEST HERE:  ');
         writeln(randc:10:3)
       end;
     if (bias >= 950) and (bias < 1300)
       then begin
         GPWHCEP(sigx, sigy, mux, muy);
         write('THE GRUBBS-PATNAIK/WILSON-HILFERTY IS BEST HERE:  ');
         writeln(gpwh:10:3)
       end
   end;
 if (ellip >= 0.6) and (ellip < 0.65) and
    (((bias >= 900) and (bias < 950)) or
     ((bias >= 1450) and (bias < 1550)))
    then begin
      GPWHCEP(sigx, sigy, mux, muy);
      write('THE GRUBBS-PATNAIK/WILSON-HILFERTY IS BEST HERE:  ');
      writeln(gpwh:10:3)
    end;
 if (ellip >= 0.65) and (ellip < 0.7) and
    (((bias >= 900) and (bias < 950)) or
     ((bias >= 1450) and (bias < 1550)))
    then begin
      RANDCEP(sigx, sigy, mux, muy);
      write('THE MODIFIED RAND-234 IS BEST HERE:  ');
      writeln(randc:10:3)
    end;
```

{BESTALGEBRAIC continued}

```
 if (ellip >= 0.7) and (ellip < 0.8)
   then begin
     if (bias < 100) or
        ((bias >= 950) and (bias < 1050)) or
        ((bias >= 1450) and (bias < 1550))
        then begin
          RANDCEP(sigx, sigy, mux, muy);
          write('THE MODIFIED RAND-234 IS BEST HERE:  ');
          writeln(randc:10:3)
        end;
     if ((bias >= 100) and (bias < 950)) or
        ((bias >= 1300) and (bias < 1450))
        then begin
          GPCSCEP(sigx, sigy, mux, muy, rho);
          write('THE GRUBBS-PATNAIK/CHI-SQUARE IS BEST HERE:  ');
          writeln(gpcs:10:3)
        end;
     if (bias >= 1050) and (bias < 1300)
        then begin
          GPWHCEP(sigx, sigy, mux, muy);
          write('THE GRUBBS-PATNAIK/WILSON-HILFERTY IS BEST HERE:  ');
          writeln(gpwh:10:3)
        end
   end;
 if (ellip >= 0.8) and (ellip < 0.9)
   then begin
     if ((bias >= 100) and (bias < 600)) or
        ((bias >= 750) and (bias < 1050)) or
        ((bias >= 1300) and (bias < 1450))
        then begin
          GPCSCEP(sigx, sigy, mux, muy, rho);
          write('THE GRUBBS-PATNAIK/CHI-SQUARE IS BEST HERE:  ');
          writeln(gpcs:10:3)
        end;
     if (bias >= 1200) and (bias < 1300)
        then begin
          GPWHCEP(sigx, sigy, mux, muy);
          write('THE GRUBBS-PATNAIK/WILSON-HILFERTY IS BEST HERE:  ');
          writeln(gpwh:10:3)
        end;
     if (bias >= 1450) and (bias < 1550)
        then begin
          RANDCEP(sigx, sigy, mux, muy);
          write('THE MODIFIED RAND-234 IS BEST HERE:  ');
          writeln(randc:10:3)
        end;
   end;
```

{BESTALGEBRAIC continued}

```
if (ellip >= 0.8) and (ellip < 0.85)
  then begin
    if (bias < 100)
      then begin
        RANDCEP(sigx, sigy, mux, muy);
        write('THE MODIFIED RAND-234 IS BEST HERE:  ');
        writeln(randc:10:3)
      end;
    if ((bias >= 600) and (bias < 950)) or
      ((bias >= 1050) and (bias < 1200))
      then begin
        GPCSCEP(sigx, sigy, mux, muy, rho);
        write('THE GRUBBS-PATNAIK/CHI-SQUARE IS BEST HERE:  ');
        writeln(gpcs:10:3)
      end
  end;
if (ellip >= 0.85) and (ellip < 0.9)
  then begin
    if (bias < 100)
      then begin
        GPCSCEP(sigx, sigy, mux, muy, rho);
        write('THE GRUBBS-PATNAIK/CHI-SQUARE IS BEST HERE:  ');
        writeln(gpcs:10:3)
      end;
    if ((bias >= 600) and (bias < 750)) or
      ((bias >= 1050) and (bias < 1200))
      then begin
        RANDCEP(sigx, sigy, mux, muy);
        write('THE MODIFIED RAND-234 IS BEST HERE:  ');
        writeln(randc:10:3)
      end
  end;
if (ellip >= 0.9) and (ellip < 0.95)
  then begin
    if (bias < 600) or
      ((bias >= 900) and (bias < 1050)) or
      ((bias >= 1250) and (bias < 1450))
      then begin
        GPCSCEP(sigx, sigy, mux, muy, rho);
        write('THE GRUBBS-PATNAIK/CHI-SQUARE IS BEST HERE:  ');
        writeln(gpcs:10:3)
      end;
    if ((bias >= 600) and (bias < 900)) or
      ((bias >= 1050) and (bias < 1250)) or
      ((bias >= 1450) and (bias < 1550))
      then begin
        RANDCEP(sigx, sigy, mux, muy);
        write('THE MODIFIED RAND-234 IS BEST HERE:  ');
        writeln(randc:10:3)
      end
  end;
```

```
{BESTALGEBRAIC continued}


 if (ellip >= 0.95) and (ellip <= 1.0)
   then begin
     if (bias < 100) or
        ((bias >= 200) and (bias < 500)) or
        ((bias >= 750) and (bias < 1200)) or
        ((bias >= 1300) and (bias < 1450))
        then begin
          GPCSCEP(sigx, sigy, mux, muy, rho);
          write('THE GRUBBS-PATNAIK/CHI-SQUARE IS BEST HERE:  ');
          writeln(gpcs:10:3)
        end;
     if ((bias >= 100) and (bias < 200)) or
        ((bias >= 500) and (bias < 750)) or
        ((bias >= 1450) and (bias < 1550))
        then begin
          RANDCEP(sigx, sigy, mux, muy);
          write('THE MODIFIED RAND-234 IS BEST HERE:  ');
          writeln(randc:10:3)
        end;
     if (bias >= 1200) and (bias < 1300)
        then begin
          GPWHCEP(sigx, sigy, mux, muy);
          write('THE GRUBBS-PATNAIK/WILSON-HILFERTY IS BEST HERE:  ');
          writeln(gpwh:10:3)
        end
   end
end;     {BESTALGEBRAIC}
```

****

```
{The following procedure is used to execute the desired
     operation that the user decided to accomplish}
```

```
procedure  PICKMETHOD(var  pick:  integer;
                           sigx, sigy, mux, muy, rho, bias, ellip:  real);

  begin
    case pick of
      1:  begin
            writeln('HOW MANY INTERVALS DO YOU WANT THE FOR THE CBN?');
            writeln('THE MORE INTERVALS, THE MORE TIME, THE MORE ACCURATE');
            readln(ints);
            GPCSCEP(sigx, sigy, mux, muy, rho);
            GPWHCEP(sigx, sigy, mux, muy);
            RANDCEP(sigx, sigy, mux, muy);
            cep:= gpcs;
            CBNORM(sigx, sigy, mux, muy, rho, cep, ints);
            EXACT(sigx, sigy, mux, muy, cep);
            writeln('CEPs FOR ALL METHODS ARE AS FOLLOWS:');
            writeln;
            writeln('GRUBBS-PATNAIK/CHI-SQUARE:  ', gpcs:10:3);
            writeln('GRUBBS-PATNAIK/WILSON-HILFERTY:  ',  gpwh:10:3);
            writeln('MODIFIED RAND-234:  ',  randc:10:3);
            writeln('CBN:  ',  cbn:10:3);
            writeln('"EXACT":  ',  cep:10:3)
          end;
      2:  begin
            GPCSCEP(sigx, sigy, mux, muy, rho);
            write('THE CEP GIVEN BY THE GRUBBS-PATNAIK/CHI-SQUARE IS:  ');
            writeln(gpcs:10:3)
          end;
      3:  begin
            GPWHCEP(sigx, sigy, mux, muy);
            write('THE CEP GIVEN BY THE GRUBBS-PATNAIK/WILSON-HILFERTY IS:  ');
            writeln(gpwh:10:3)
          end;
      4:  begin
            RANDCEP(sigx, sigy, mux, muy);
            writeln('THE CEP GIVEN BY THE MODIFIED RAND-234 IS:  ', randc:10:3);
          end;
      5:  begin
            writeln('HOW MANY INTERVALS DO YOU WANT THE FOR THE CBN?');
            writeln('THE MORE INTERVALS, THE MORE TIME, THE MORE ACCURATE');
            readln(ints);
            CBNORM(sigx, sigy, mux, muy, rho, cep, ints);
            writeln('THE CEP GIVEN BY THE CBN IS:  ', cbn:10:3)
          end;
      6:  begin
            EXACT(sigx, sigy, mux, muy, cep);
            writeln('THE CEP GIVEN BY THE "EXACT" IS:  ', cep:10:3)
          end;
      7:  BESTALGEBRAIC(ints, sigx, sigy, mux, muy, rho, bias, ellip)
    end
  end;
```

****

```
  {The following is the Main program.  It accepts the input
       of the user for the variables and then which method
       the user would like to calculate}

begin {MAIN PROGRAM}
  writeln('PLEASE INPUT THE SAMPLE STANDARD DEVIATIONS AND MEANS');
  write('CROSSRANGE MEAN {X BAR}:  ');
  readln(mux);
  write('CROSSRANGE STANDARD DEVIATION {SIGMA X}:  ');
  readln(sigx);
  write('DOWNRANGE MEAN {Y BAR}:  ');
  readln(muy);
  write('DOWNRANGE STANDARD DEVIATION {SIGMA Y}:  ');
  readln(sigy);
  write('INPUT THE CORRELATION COEFFICIENT, IF ANY {RHO}:  ');
  readln(rho);
  bias:= sqrt(sqr(mux) + sqr(muy));
  ellip:= sigx/sigy;
  writeln('HOW WOULD YOU LIKE FOR THE PROGRAM TO RUN?');
  writeln('             {ENTER APPROPRIATE NUMBER}');
  writeln;
  writeln('1 = CALCULATE ALL METHODS');
  writeln('2 = CALCULATE GRUBBS-PATNAIK/CHI-SQUARE METHOD ONLY');
  writeln('3 = CALCULATE GRUBBS-PATNAIK/WILSON-HILFERTY METHOD ONLY');
  writeln('4 = CALCULATE MODIFIED RAND-234 METHOD ONLY');
  writeln('5 = CALCULATE CBN METHOD ONLY');
  writeln('6 = CALCULATE "EXACT" METHOD ONLY');
  writeln('7 = CALCULATE THE "BEST" OF THE ALGEBRAIC METHODS');
  readln(pick);
  writeln;
  writeln('**************************************************************');
  writeln;
  PICKMETHOD(pick, sigx, sigy, mux, muy, rho, bias, ellip);
end.
```

Bibliography

1. Astbury, K. A., Ph. D, Accuracy Analysis Section. Inter-office correspondence, 85.H235.KAA-001. Subject: Computing CEP According to the Popular Approximation Formulas. TRW Defense Systems Group, Redondo Beach CA, 29 January 1985.

2. Auclair, Capt Paul. Telephone interview. AFCSA/SASM, Washington DC, 22 April 1985.

3. Berg, Capt David H., Correspondence. Subject: "Comparison of Rotated Rand-234 and Correlated Bivariate Normal (CBN) CEP Computations." 1 STRAD/BM, Vandenberg AFB, CA, 13 July 1984.

4. Berg, Capt David H. Telephone interview. 4220 WSES/BM Offutt AFB NE, 3 May 1985.

5. <u>CRC Standard Mathematical Tables</u> 27th Ed. CRC Press, Boca Raton, FL, 1984.

6. Ethridge, Maj Ronald A., "Robust Estimation of Circular Error Probable for Small Samples," Air Command and Staff College Student Report #83-0690, Maxwell AFB, AL, 1983.

7. Grubbs, Frank E. "Approximate Circular and Noncircular Offset Probabilities of Hitting," <u>Operations Research, 12</u> (1): 51-62 (January-February 1964).

8. Harter, H. Leon. "Circular Error Probabilities," <u>Journal of the American Statistical Association, 55</u> (292): 723-731 (December 1960).

9. Hines, William C. and Douglas C. Montgomery, <u>Probability and Statistics in Engineering and Management Science</u> 2nd Ed. John Wiley and Sons, NY, NY, 1972.

10. Hodges, Jere and L. G. Allred. Air Force letter. Subject: Minuteman II CEP Methodology. MMEWM, Ogden AFB, Utah (no date available).

11. Jones, C. R., Analyst, Accuracy Analysis Section, TRW. Correspondence, 84.H235.53-003. Subject: Correction of CBN Program to Calculate CEP. TRW Defense Systems Group, Redondo Beach CA, 26 July 1984.

12. Memo for record. Subject: Development of CEP Formula. OAOC, Offutt Air Force Base Nebraska, 16 Nov 1964

13. "Offset Circle Probabilities." Report R-234. Computer Sciences Department, The RAND Corporation, Santa Monica, CA, 14 March 1952.

14. Pesapane, Capt John and Maj Robert B. Irvine Jr., "Derivation of CEP Formula to Approximate Rand-234 Tables." Ballistic Missile Evaluation, HQ SAC, Offutt AFB, NE, February 1977.

15. Simkins, L. S. Correspondence, PM-4746. Subject: Calcula-tion of Circle of Equiprobability for a Biased, Correlated, Bivariate, Normal Distribution. Applied Physics Laboratory, The Johns Hopkins University, Silver Spring Maryland, 26 March 1974.

16. Smith, C. C. Project Engineer, Minuteman Q. C. Data Base. Interoffice correspondence, 82.F651.24-008. Subject: Methods of CEP Calculation. TRW Defense and Space Systems Group, Redondo Beach CA, 26 July 1982.

17. Smith, C. C., Project Engineer, Minuteman Q. C. Data Base. Interoffice correspondence, 84.H235.36-004. Subject: CEP Calculation Using Infinite Series. TRW Defense Systems Group, Redondo Beach CA, 31 March 1984.

18. Tucker, Shirley J. and James R. Thacker, Guidance and Controll Directorate, U.S. Army Missile Laboratory. "CEP Tabular Estimation for Biased and Elliptical Distributions." Technical Report RG-84-7. U.S. Army Missile Command, Redstone Arsenal, AL, December 1983.

VITA

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